GENERALIZED STAIRCASES: RECURRENCE AND SYMMETRY

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Abstract. We study infinite translation surfaces which are $\mathbb{Z}$-covers of compact translation surfaces. We obtain conditions ensuring that such surfaces have Veech groups which are Fuchsian of the first kind and give a necessary and sufficient condition for recurrence of their straight-line flows. Extending results of Hubert and Schmithüsen, we provide examples of infinite non-arithmetic lattice surfaces, as well as surfaces with infinitely generated Veech groups.

1. Introduction

The geometry of translation surfaces has been intensively studied in recent years (see [MT02] and [Zor06] for definitions and a survey of recent work). While most of the work was concerned with compact surfaces, in several recent papers non-compact surfaces were also considered. For instance, in [CGL06], the horseshoe and baker's transformations were realized by an affine transformation; [Hoo08] is a study of the geometry and dynamics of an infinite translation surface which arises as a geometric limit of compact lattice surfaces; in [HW08], a connection was made to $\mathbb{Z}$-valued skew products over 1-dimensional systems, and in [Val], the topology of the unfolding surface for an irrational billiard was determined. Removing the restriction that the surface is compact gives a flexible setup and many phenomena, absent in the compact case, may be observed. For example, in the recent paper [HS09], Hubert and Schmithüsen made the surprising discovery that there are infinite square tiled surfaces whose Veech group is infinitely generated.

The examples studied in [HW08, HS09] are $\mathbb{Z}$-covers of compact translation surfaces. Although this class is much smaller than the general case, it already displays many surprising features. It may be hoped that it provides a good starting point for a study of the geometry and dynamics of infinite translation surfaces. In this paper we begin the systematic study of these surfaces. Our analysis yields a bijection between $\mathbb{Z}$-covers $\widetilde{M} \to M$, ramified over a finite set $P \subset M$, and projective classes of elements $w \in H_1(M, P; \mathbb{Z})$ (Proposition 7). Under this bijection, *recurrent $\mathbb{Z}$-covers*, i.e. covers on which the straightline flow is recurrent in almost all directions, correspond to homology classes with vanishing holonomy (Proposition 15). Utilizing a theorem of Thurston which appeared in the unpublished manuscript [Thu98], we obtain a sufficient condition ensuring that the Veech group of a cover $\widetilde{M}$ is Fuchsian of the first kind (Theorem 22). This result implies that any recurrent $\mathbb{Z}$-cover of a square tiled surface in genus 2 has a Veech group which is of the first kind (Corollary 23), extending the results of [HS09]. We also obtain necessary and sufficient conditions for a (finite power of a) parabolic element in the Veech group of M to lift to the Veech group of every recurrent $\mathbb{Z}$-cover $\widetilde{M}$ (Theorem 27). Using it one may reprove some of the results of [HS09] in a more general setting. We illustrate the use of our results in the last section, where we

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provide an example of an infinite lattice translation surface with a non-arithmetic Veech group (Proposition 31), and answer a question of Hubert and Schmithüsen (Section 7.2).

2. Regular covers of translation surfaces

Let M denote a compact translation surface and $P \subset M$ denote a finite (possibly empty) subset. We consider P to be a collection of punctures of the surface M and will use M° to denote $M \smallsetminus P$.

Recall that the translation surface M° comes equipped with local charts to $\mathbb{R}^2$ defined away from a discrete set of singularities, such that the transition functions are all translations [MT02] [Zor06]. An *affine automorphism* of M° is a homeomorphism $f : M^\circ \to M^\circ$ which preserves the underlying affine structure of M°. The local charts identify the tangent plane $T_P M^\circ$ of every non singular point P with the plane $T_{\mathbf{0}}\mathbb{R}^2 = \mathbb{R}^2$. If f is an affine automorphism, then the induced actions on the tangent planes $T_P M^\circ \to T_{f(P)} M^\circ$, as identified with $\mathbb{R}^2$, are the same. We call this induced map *the derivative of* f, $D(f) : \mathbb{R}^2 \to \mathbb{R}^2$. Note that $D(f) \in GL(2, \mathbb{R})$, and if M is compact then $D(f)$ has determinant ± 1. The collection of all affine automorphisms of M° forms the *affine automorphism group* $Aff(M^\circ)$. The group $\Gamma(M^\circ) = D\big(Aff(M^\circ)\big) \subset GL(2, \mathbb{R})$ is called the *Veech group* of M°.

Covering space theory associates covers of a space with the subgroups of its fundamental group. A cover is called *regular* if it is associated to a normal subgroup. We consider a normal subgroup $N \subset \pi_1(M^\circ)$, and consider the associated cover $\pi : \widetilde{M} \to M^\circ$. The group $\Delta = \pi_1(M^\circ)/N$ acts on $\widetilde{M}$ as the automorphisms of the cover, with $\widetilde{M}/\Delta = M^\circ$.

We have the following from covering space theory.

Proposition 1.

(1) *An element $f \in Aff(M^\circ)$ lifts to an $\tilde{f} \in Aff(\widetilde{M})$ if and only if $f_*(N) = N$.*

(2) *An element $\tilde{f} \in Aff(\widetilde{M})$ descends to an $f \in Aff(M^\circ)$ if and only if $\tilde{f}$ normalizes the deck group Δ. That is, $\tilde{f}\Delta\tilde{f}^{-1} = \Delta$.*

Definition 2. The *affine automorphism group of a cover* $\widetilde{M} \to M^\circ$ is the group of pairs of elements $(\tilde{f}, f) \in Aff(\widetilde{M}) \times Aff(M^\circ)$ for which $\pi \circ \tilde{f} = f \circ \pi$. We denote this group by $Aff(\widetilde{M}, M^\circ)$. A necessary condition for $(\tilde{f}, f) \in Aff(\widetilde{M}, M^\circ)$ is that $D(\tilde{f}) = D(f)$. Thus we have a canonical definition of the derivative $D : Aff(\widetilde{M}, M^\circ) \to GL(2, \mathbb{R})$. We call the image of the group homomorphism D the *Veech group of the cover*, and denote it by $\Gamma(\widetilde{M})$.

Let $G_N = \{f \in Aff(M^\circ) \ : \ f_*(N) = N\}$. For an $f \in G_N$ the action of f_* on $\pi_1(M^\circ)$ induces an action on $\Delta = \pi_1(M^\circ)/N$. The following is an immediate consequence:

Corollary 3. *$Aff(\widetilde{M}, M^\circ) \cong \Delta \rtimes G_N$, with G_N acting on Δ as mentioned above. Indeed, we have a short exact sequence*

$$1 \to \Delta \hookrightarrow Aff(\widetilde{M}, M^\circ) \twoheadrightarrow G_N \to 1.$$

□

Note that the projection $p : Aff(\widetilde{M}, M) \to Aff(\widetilde{M})$ may not be injective. However, we do not miss much.

Proposition 4. *If M° is not an unpunctured torus, then $p\big(Aff(\widetilde{M}, M^\circ)\big)$ is a finite index subgroup of $Aff(\widetilde{M})$.*

Proof. Consider the group $\mathcal{T} \subset Aff(\widetilde{M})$ of elements ι for which $D(\iota) = I$, i.e. the group of translation automorphisms of $\widetilde{M}$. We claim that $\mathcal{T}$ acts properly discontinuously on the set of non-singular points of $\widetilde{M}$. To see this, let Q denote the union of the singularities of M with P. By assumption Q is non-empty. The surface M° has a Delaunay decomposition relative to the points in Q. See [MS91, §1] for background. The Delaunay decomposition of $\widetilde{M}$ relative to the lifts of Q is the lift of the decomposition of M°. A translation automorphism must permute the cells in the Delaunay decomposition, and hence is properly discontinuous.

The deck group Δ is a finite index subgroup of $\mathcal{T}$, because $\text{Area}(\widetilde{M}/\Delta) = \text{Area}(M) < \infty$. The group Δ is finitely generated because it is a quotient of $\pi_1(M^\circ)$, which is finitely generated. $\mathcal{T}$ is also finitely generated as it contains Δ as a finite index subgroup. An element $\widetilde{f} \in Aff(\widetilde{M})$ acts on $\mathcal{T}$ by conjugation, and preserves the index of subgroups. There are only finitely many subgroups of $\mathcal{T}$ with index $[\mathcal{T} : \Delta]$, because $\mathcal{T}$ is finitely generated. Thus, a finite index subgroup of $Aff(\widetilde{M})$ normalizes Δ. The conclusion follows by Proposition 1. □

Presumably, nearly every countable subgroup of $GL(2, \mathbb{R})$ arises as a Veech group of some infinite translation surface. (See [PSV09] for an investigation of Veech groups of tame translation surfaces homeomorphic to the Loch Ness monster.) However, because Veech groups of compact translation surfaces are discrete [Vee89], we have different answer for normal covers.

Corollary 5. *If M° is not an unpunctured torus, the Veech group $\Gamma(\widetilde{M})$ is a discrete subgroup of $\widehat{SL}(2, \mathbb{R})$, the group of 2×2 real matrices of determinant ± 1.*

3. $\mathbb{Z}$-covers

We use $H_1(M, P; \mathbb{Z})$ to denote the relative homology of M with respect to the set of punctures, and $H_1(M^\circ; \mathbb{Z})$ denotes the absolute homology of the punctured surface. Intersection number is a non-degenerate bilinear form

$$i : H_1(M, P; \mathbb{Z}) \times H_1(M^\circ; \mathbb{Z}) \to \mathbb{Z}.$$

Definition 6. The $\mathbb{Z}$*-cover of M° associated to a non-zero* $w \in H_1(M, P; \mathbb{Z})$ is the cover associated to the kernel of the homomorphism

$$\varphi_w : \pi_1(M^\circ) \to \mathbb{Z}, \quad \gamma \mapsto i(w, [\![\gamma]\!]),$$

where $[\![\gamma]\!]$ denotes the homology class of γ. We denote this cover by $\widetilde{M}_w$.

If A is a free abelian group, we use $\mathbb{P}A$ to denote $(A \smallsetminus \{\mathbf{0}\})/\sim$, where $a \sim b$ if there are non-zero $m, n \in \mathbb{Z}$ for which $ma = nb$. By non-degeneracy of the bilinear intersection form, we have:

Proposition 7. *The $\mathbb{Z}$-covers $\widetilde{M}_w$ and $\widetilde{M}_{w'}$ are the same if and only if $w \sim w'$.*

Thus, the space of $\mathbb{Z}$-covers of M° is naturally identified with $\mathbb{P}H_1(M, P; \mathbb{Z})$. Statement (1) of proposition 1 can be restated as follows.

Proposition 8. *An $f \in Aff(M^\circ)$ lifts to an $\widetilde{f} \in Aff(\widetilde{M}_w)$ if and only if $f_*(w) = \pm w$, where f_* denotes the action of f on $H_1(M, P; \mathbb{Z})$.*

We conclude this section with some remarks on the topology of $\mathbb{Z}$-covers. Since we will not be using these results in the sequel, they will be stated without proof.

Loosely speaking, we think of $\widetilde{M}_w$ as a cover of M ramified over points of P. To make this intuition precise, recall that by pulling back the Euclidean metric, we may endow a translation surface with a metric, and consider its completion. The completion of M° is M, and the map π extends to a map $\bar{\pi} : \bar{M} \to M$, where $\bar{M}$ is the completion of $\widetilde{M}_w$. It is natural to inquire whether $\bar{\pi}$ is a covering map. To this end we have:

Proposition 9. *For each $p \in P$, let $U_p \subset M$ be an open disk with boundary curve γ_p such that $U_p \cap P = \{p\}$ and $\gamma_p \cap P = \varnothing$. Let $\bar{U}_p = \bar{\pi}^{-1}(U_p)$. Then $\bar{\pi}|_{\bar{U}_p}$ is a covering map if and only if $i(w, [\![\gamma_p]\!]) = 0$.*

Thus, $\bar{\pi}$ is a covering map if and only if $i(w, [\![\gamma_p]\!]) = 0$ for all $p \in P$. The following is equivalent.

Corollary 10. *The map $\bar{\pi}$ is a covering map if and only if w is an element of $H_1(M; \mathbb{Z})$, viewed as a subset of $H_1(M, P; \mathbb{Z})$.*

In case $i(w, [\![\gamma_p]\!]) \neq 0$, there is no multiple of γ_p which lifts to $\widetilde{M}_w$ as a closed loop. In this case we call any $\bar{p} \in \bar{\pi}^{-1}(p)$ an *infinite singularity*, since the map $\bar{\pi} : \bar{M} \to M$ in a neighborhood of $\bar{p}$ has the structure of an 'infinite cone angle singularity' or a 'logarithmic singularity'. To compute the number of such points on $\bar{M}$, we have:

Proposition 11. *Assume w is a primitive element of $H_1(M, P; \mathbb{Z})$. Suppose $p \in P$ is such that $i(w, [\![\gamma_p]\!]) \neq 0$. Then $|\bar{\pi}^{-1}(p)| = |i(w, [\![\gamma_p]\!])|$. In particular the number of infinite singularities is finite.*

If $\bar{M}$ has an infinite singularity $\bar{p}$, its metric topology is not proper. Indeed, the closure of any small ball around $\bar{p}$ is not compact. Therefore it is natural to consider $\widehat{M}$, the complement in $\bar{M}$ of the infinite singularities. That is, $\widehat{M}$ is the largest subset of $\bar{M}$ such that the restriction of $\bar{\pi}$ to $\widehat{M}$ is a covering map. Repeating the arguments of [Val] and recalling terminology of [Ric63], we may understand the topology of $\widehat{M}$. We have:

Proposition 12. *If $\bar{M}$ has infinite singularities, then $\widehat{M}$ has only one end and is in the homeomorphism class of the 'Loch Ness monster', the orientable infinite genus surface with a single end. If $\bar{M}$ has no infinite singularities then it has two ends. In this case, $\bar{M}$ is either a cylinder or is homeomorphic to the orientable infinite genus surface with two non-planar ends.*

4. Recurrent $\mathbb{Z}$-covers

A translation surface has a *holonomy map* $\mathbf{hol} : H_1(M, P; \mathbb{Z}) \to \mathbb{R}^2$, obtained by developing a representative of the class into $\mathbb{R}^2$ and taking the difference of the starting and end points. For dynamical reasons, we are especially interested in $\mathbb{Z}$-covers with the following property.

Definition 13 (Recurrent $\mathbb{Z}$-covers)**.** The $\mathbb{Z}$-cover $\widetilde{M}_w$ is called *recurrent* if $\mathbf{hol}(w) = \mathbf{0}$.

Although not explicitly stated, square-tiled covers of this type were studied before in [HW08] and [HS09]. One reason for restricting attention to recurrent $\mathbb{Z}$-covers is that non-recurrent $\mathbb{Z}$-covers have few affine symmetries. A discrete subgroup of $\widehat{SL}(2, \mathbb{R})$ is called *elementary* if it contains a finite index abelian subgroup and *non-elementary* otherwise. Conversely, a non-elementary subgroup of $\widehat{SL}(2, \mathbb{R})$ contains the free group with two generators [MT98, Theorem 2.9]. We have the following corollary of Proposition 8.

Corollary 14. *If there is an $A \in \Gamma(\widetilde{M}_w) \cap SL(2,\mathbb{R})$ with* $\mathrm{trace}(A) \neq \pm 2$*, then $\widetilde{M}_w$ is a recurrent $\mathbb{Z}$-cover. In particular, if $\Gamma(\widetilde{M}_w)$ is non-elementary, then $\widetilde{M}_w$ is a recurrent $\mathbb{Z}$-cover.*

Proof. We prove the contrapositive. Suppose $\mathbf{hol}(w) \neq \mathbf{0}$. By Proposition 8, $(\widetilde{f}, f) \in (\widetilde{M}_w, M^\circ)$ implies that $f_*(w) = \pm w$. Then $D(f)\big(\mathbf{hol}(w)\big) = \mathbf{hol}\big(f_*(w)\big) = \pm\mathbf{hol}(w)$. Thus,

$$\Gamma(\widetilde{M}_w) \subset \Big\{A \in \widehat{SL}(2,\mathbb{R}) \ : \ A\big(\mathbf{hol}(w)\big) = \pm\mathbf{hol}(w)\Big\} \cong (\mathbb{R} \rtimes \mathbb{Z}/2\mathbb{Z}) \oplus \mathbb{Z}/2\mathbb{Z}.$$

We conclude $\Gamma(\widetilde{M}_w)$ is abelian or contains an index two abelian subgroup. Moreover, all elements $A \in \Gamma(\widetilde{M}_w) \cap SL(2,\mathbb{R})$ have trace ± 2. □

We will now justify the term recurrent $\mathbb{Z}$-cover. Let $F_t^\theta : M \to M$ denote the straight-line flow in direction $\theta \in S^1$. Similarly, we will use $\widetilde{F}_t^\theta : \widetilde{M} \to \widetilde{M}$ to denote the straight-line flow on a $\mathbb{Z}$-cover $\widetilde{M}$ in direction θ. Recall that a measure preserving flow F_t is called *recurrent* if for any measurable set A, for a.e. $x \in A$ there is $t_n \to \infty$ such that $F_{t_n}x \in A$.

Proposition 15 (Recurrence of the straight-line flow)**.** *Let $\widetilde{M}$ be a $\mathbb{Z}$-cover of M°. Then $\widetilde{M}$ is a recurrent $\mathbb{Z}$-cover if and only if for any θ for which F_t^θ is ergodic, $\widetilde{F}_t^\theta$ is recurrent.*

Proof. We will reduce the statement to a classical result of K. Schmidt [Sch77, Theorem 11.4] in infinite ergodic theory. Suppose (X, μ) is a finite measure space and $T : X \to X$ is a measurable transformation preserving μ which is ergodic. For a measurable $f : X \to \mathbb{Z}$, $f \in L^1(X, \mu)$, define $X_f = X \times \mathbb{Z}$ and

$$T_f : X_f \to X_f, \quad T_f(x,k) = (Tx, k + f(x))\,.$$

Then T_f is recurrent if and only if $\int f \, d\mu = 0$.

Given θ, we reduce to the above statement as follows: choose a segment α in M, which is in the direction θ' perpendicular to θ. Define $\widetilde{\alpha}$ in $\widetilde{M}$ to be the union of all lifts of α to $\widetilde{M}$. Denote by T (resp. $\widetilde{T}$) the Poincaré return map to the section α (resp. $\widetilde{\alpha}$), so that T is an interval exchange transformation. The ergodicity of T is equivalent to that of F_t^θ and the recurrence of $\widetilde{F}_t^\theta$ is equivalent to the recurrence of $\widetilde{T}$. Since continuous maps have Borel sections, we may (measurably) identify $\widetilde{M}$ with $M \times \mathbb{Z}$. In these coordinates $\widetilde{T} = T_f$ where

$$f = f^{(\theta)} : \alpha \to \mathbb{Z}, \quad f(x) = i(w, [\![\beta_x]\!]),$$

and β_x is the curve from x to Tx along the F_t^θ orbit of x, and then from Tx to x along α. Let μ be the length measure on α. Up to scaling, Lebesgue measure on M can be represented as $d\mu dt$, where dt denotes the length measure along the orbits of F_t^θ. Since f assumes finitely many values, one on each interval of continuity of T, it is in $L^1(\alpha, \mu)$.

Label by $I_1, \ldots, I_\ell$ be the partition of α into intervals of continuity for T. By refining this decomposition we assume that the flow in direction θ starting from the interior of I_j does not hit a puncture in P. For each j, let β_j be a closed loop β_{x_j} as above, corresponding to some $x_j \in I_j$; the particular choice of x_j does not affect $[\![\beta_j]\!]$. Now write $\beta = \sum \mu(I_j)[\![\beta_j]\!] \in H_1(M;\mathbb{R})$. We claim that for a path γ on M representing an element of $H_1(M, P; \mathbb{Z})$,

$$i(\gamma, [\![\beta]\!]) = \mathbf{hol}_{\theta'}(\gamma),$$

i.e. the holonomy vector orthogonally projected onto the one-dimensional vector space in direction θ'. Indeed after homotoping γ off α, each positive crossing of ℓ_j means γ has crossed the rectangle above I_j, and contributes $\mu(I_j)$ to $\mathbf{hol}_{\theta'}(\gamma)$. Therefore

$$\int f^{(\theta)}\, d\mu = \sum_j \mu(I_j) i(w, [\![\gamma_j]\!]) \\ = i(w, [\![\beta]\!]) = \mathbf{hol}_{\theta'}(w).$$

The main theorem of [KMS85] guarantees the existence of two independent ergodic θ. We see that $\int f^{(\theta)}\, d\mu = 0$ for any ergodic direction θ on M, if and only if $\mathbf{hol}(w) = 0$. □

A similar argument was employed by Conze and Gutkin in [CG10] to prove recurrence of the billiard flow on some infinite billiard tables.

Corollary 16. *If* $\mathbf{hol}(w) = 0$*, the straightline flow* $\widetilde{F}_t^\theta$ *on* $\widetilde{M}_w$ *is recurrent for a.e.* θ*.*

Proof. Combine Proposition 15 with the famous result of Kerckhoff, Masur and Smillie [KMS85]. □

5. Veech groups of recurrent $\mathbb{Z}$-covers

Let $H \subset \mathbb{R}^2$. We define $K(H)$ to be the smallest extension field of $\mathbb{Q}$ for which there is an $A \in GL(2, \mathbb{R})$ such that $A(H) \subset K(H)^2 \subset \mathbb{R}^2$. The *holonomy field* of a translation surface M is the field $k = K\big(\mathbf{hol}(H_1(M; \mathbb{Z}))\big)$. The holonomy field was first introduced and studied by Gutkin and Judge [GJ00]. We will follow the treatment of the holonomy field given in the appendix of [KS00]. It is known (see [KS00], Theorem 28) that if M is compact and there is a pseudo-Anosov homeomorphism in $\mathit{Aff}(M)$, then k is a field extension of $\mathbb{Q}$ of degree at most the genus g of M. Moreover, the image $\mathbf{hol}\big(H_1(M; \mathbb{Z})\big)$ is a $\mathbb{Z}$-module of rank $2[k : \mathbb{Q}]$.

It follows from the work of Kenyon and Smillie that if $\mathit{Aff}(M^\circ)$ contains a pseudo-Anosov homeomorphism then $K\big(\mathbf{hol}(H_1(M; \mathbb{Z}))\big) = K\big(\mathbf{hol}(H_1(M, P; \mathbb{Z}))\big)$. We unambiguously declare this the holonomy field in this case, and we use k to denote this field.

Definition 17 (Holonomy-free subspaces)**.** The *holonomy-free subspaces of homology* are $W = \mathit{ker}\ \mathbf{hol} \subset H_1(M, P; \mathbb{Z})$ of relative homology, and $W_0 = W \cap H_1(M; \mathbb{Z})$ of absolute homology.

The $\mathbb{Z}$-modules W_0 and W have ranks given by the following equations.

$$\text{rk } W_0 = \text{rk } H_1(M; \mathbb{Z}) - 2[k : \mathbb{Q}] = 2(g - [k : \mathbb{Q}]).$$

$$\text{rk } W = \text{rk } H_1(M, P; \mathbb{Z}) - 2[k : \mathbb{Q}] = \begin{cases} 2(g - [k : \mathbb{Q}]) + \#P - 1 & \text{if } P \neq \emptyset \\ 2(g - [k : \mathbb{Q}]) & \text{otherwise.} \end{cases}$$

The affine automorphism group $\mathit{Aff}(M^\circ)$ acts on homology and preserves the subspaces W_0 and W. Thus, we have the following group homomorphisms.

$$\psi_0 : \mathit{Aff}(M^\circ) \to \mathit{Aut}(W_0), \quad f \mapsto f_*|_{W_0}.$$

$$\psi : \mathit{Aff}(M^\circ) \to \mathit{Aut}(W), \quad f \mapsto f_*|_{W}.$$

The following statement follows immediately from Proposition 8. It explains our interest in these homomorphisms.

Proposition 18. *Let $f \in ker\ \psi$. For each $w \in W$, there is an $\widetilde{f} \in Aff(\widetilde{M}_w)$ such that $(\widetilde{f}, f) \in Aff(\widetilde{M}_w, M^\circ)$. The subgroup*

$$\{(\widetilde{f}, f) \in Aff(\widetilde{M}_w, M^\circ) \ : \ f \in ker\ \psi\}$$

is normal inside $Aff(\widetilde{M}_w, M^\circ)$.

The elements of $Aff(M^\circ)$ permute the punctures. Let $\rho : Aff(M^\circ) \to Sym(P)$ be the map which assigns to an $f \in Aff(M^\circ)$ the permutation induced on P. We have the following.

Proposition 19. *$\psi(ker\ \psi_0 \cap ker\ \rho)$ is abelian of rank at most* $(\mathrm{rk}\ W_0)(\mathrm{rk}\ W - \mathrm{rk}\ W_0)$. *Thus, there is an exact sequence*

$$1 \to ker\ \psi \hookrightarrow ker\ \psi_0 \twoheadrightarrow A \to 1$$

where $A \subset \mathbb{Z}^{(\mathrm{rk}\ W_0)(\mathrm{rk}\ W - \mathrm{rk}\ W_0)} \rtimes Sym(P)$ has a finite index free abelian subgroup.

Proof. Enumerate $P = \{p_1, \ldots, p_n\}$, and let $\gamma_i \in H_1(M^\circ; \mathbb{Z})$ be the homology class of a loop which travels clockwise around p_i for $i = 1, \ldots, n$. Let $J : W \to \mathbb{Z}^n$ denote the function

$$J(w) = \big(i(w, \gamma_1), \ldots, i(w, \gamma_n)\big) \in \mathbb{Z}^n.$$

Note that for all $f \in Aff(M^\circ)$ we have $J \circ f_*(w) = \rho(f) \circ J(w)$, where the permutation $\rho(f)$ is acting as a permutation matrix. In addition, $J(w)$ determines the coset of W/W_0 which contains w. The following statements follow from this discussion.

(1) $ker\ J = W_0$.
(2) If $f \in ker\ \rho$, then $f_*(w) - w \in W_0$ for all $w \in W$.

By definition, if $f \in ker\ \psi_0$, then $f_*(w_0) = w_0$ for all $w_0 \in W_0$. For $f \in ker\ \psi_0 \cap ker\ \rho$, let $h_f : W/W_0 \to W_0$ denote the map $w + W_0 \mapsto f_*(w) - w$. This is well defined by the above discussion. Moreover, we can recover $\psi(f) = f_*|_W$ via the formula $\psi(f)(w) = w + h_f(w + W_0)$. If $f, g \in ker\ \psi_0 \cap ker\ \rho$,

$$\begin{aligned} \psi(g \circ f)(w) = \psi(g)\big(w + h_f(w + W_0)\big) &= w + h_f(w + W_0) + h_g\big(w + h_f(w + W_0) + W_0\big) \\ &= w + h_f(w + W_0) + h_g(w + W_0). \end{aligned}$$

So $\psi(ker\ \psi_0 \cap ker\ \rho)$ is abelian group. Moreover, an element $\psi(f)$ of this group is uniquely determined by the linear map $h_f : W/W_0 \to W_0$. It can be observed that $W/W_0 \cong \mathbb{Z}^{\mathrm{rk}\ W - \mathrm{rk}\ W_0}$ and $W_0 \cong \mathbb{Z}^{\mathrm{rk}\ W_0}$. Hence, the space of all possible h_f is isomorphic to $\mathbb{Z}^{(\mathrm{rk}\ W_0)(\mathrm{rk}\ W - \mathrm{rk}\ W_0)}$. □

If G is a discrete subgroup of $GL(2, \mathbb{R})$, we will use $\Lambda G \subset \mathbb{RP}^1$ to denote the limit set of the projection of G to $PGL(2, \mathbb{R}) = \mathrm{Isom}(\mathbb{H}^2)$. A subgroup G of $GL(2, \mathbb{R})$ or $PGL(2, \mathbb{R})$ is elementary if and only if ΛG contains two or fewer points. See [MT98] for background on the limit set and for the following.

Lemma 20 (Limit sets of normal subgroups)**.** *Suppose G is a non-elementary discrete subgroup of $GL(2, \mathbb{R})$ or $PGL(2, \mathbb{R})$. If N is a non-trivial normal subgroup of G, then $\Lambda N = \Lambda G$.*

Theorem 21. *If $D\big(Aff(M^\circ)\big)$ is non-elementary and $D(ker\ \psi_0)$ is non-trivial, then*

$$\Lambda D\big(Aff(M^\circ)\big) = \Lambda D(ker\ \psi_0) = \Lambda D(ker\ \psi).$$

In this case, $\Lambda\Gamma(\widetilde{M}_w) = \Lambda D\big(Aff(M^\circ)\big)$ for all recurrent $\mathbb{Z}$-covers $\widetilde{M}_w$ of M°.

Proof. If $D(ker\ \psi_0)$ is non-trivial, then by a direct application of Lemma 20, $\Lambda D\big(Aff(M^\circ)\big) = \Lambda D(ker\ \psi_0)$. In particular, $D(ker\ \psi_0)$ is non-elementary and thus contains a free group with two generators [MT98, Theorem 2.9]. By Proposition 19, $D(ker\ \psi)$ is a finite index subgroup of the kernel of a map from $D(ker\ \psi_0)$ to an abelian group. Hence, $D(ker\ \psi)$ is non-empty. By another application of Lemma 20, we see $\Lambda D(ker\ \psi) = \Lambda D(ker\ \psi_0)$. □

A *Fuchsian group of the first kind* is a discrete subgroup Γ of $\mathrm{Isom}(\mathbb{H}^2)$ (or some other linear group which acts isometrically on $\mathbb{H}^2$) for which $\Lambda\Gamma = \mathbb{RP}^1$.

Theorem 22. *Suppose $D\big(Aff(M^\circ)\big)$ is a lattice and that* $\mathrm{rk}\ W_0 \leq 2$*. Then $D(ker\ \psi)$ is a Fuchsian group of the first kind. In particular, for any $w \in W$, $\Gamma(\widetilde{M}_w)$ is Fuchsian of the first kind.*

Proof. By Theorem 21, it is sufficient to show that $D(ker\ \psi_0)$ is non-trivial. Note that $\mathrm{rk}\ W_0$ is even. If $\mathrm{rk}\ W_0 = 0$, then $ker\ \psi_0 = Aff(M^\circ)$. The more difficult case is when $\mathrm{rk}\ W_0 = 2$. We will assume that $ker\ \psi_0$ is empty and derive a contradiction.

By the Selberg lemma, the group $D\big(Aff(M^\circ)\big)$ contains a finite index subgroup Γ which is torsion free [MT98, Theorem 2.29]. As observed by Veech [Vee89], $\mathbb{H}^2/D\big(Aff(M^\circ)\big)$ is not co-compact. Therefore, Γ is isomorphic to the fundamental group of the punctured surface $\mathbb{H}^2/\Gamma$, which is a free group. This free group Γ pulls back to a free group $F \subset Aff(M^\circ)$ such that $D|_F$ is injective.

Since $\mathrm{rk}\ W_0 = 2$, $\psi_0 : F \to \widehat{SL}(2,\mathbb{Z})$, where $\widehat{SL}(2,\mathbb{Z})$ denotes the set of 2×2 matrices of determinant ± 1. By our assumption from the first paragraph, $\psi_0|_F$ is injective. Without loss of generality, we may assume that $\psi_0(F) \subset SL(2,\mathbb{Z})$. (If not, replace F by the index two subgroup for which this is true.)

Summarizing the previous two paragraphs, we have two faithful representations, $D|_F$ and $\psi_0|_F$, of F into $SL(2,\mathbb{R})$. We will derive a contradiction from properties of these representations. These representations satisfy the following statements for all $f \in F$.

(1) If $D(f)$ is parabolic, then $\psi_0(f)$ is also parabolic.
(2) If $D(f)$ is hyperbolic, then $2 \leq |\mathrm{tr}\ \psi_0(f)| < |\mathrm{tr}\ D(f)|$.

Statement 1 is true because if $f \in Aff(M^\circ)$ is a parabolic, then some power of f is a multi-twist of M°. All eigenvalues of the action of a multi-twist on homology are 1. In particular, the eigenvalues for the action of f on homology are all of modulus 1. Thus, $\psi_0(f)$ is either elliptic or parabolic. But, if $\psi_0(f)$ is elliptic, then ψ_0 is not faithful. If $D(f)$ is hyperbolic, then $f \in Aff(M^\circ)$ is a pseudo-Anosov homeomorphism. Let λ be the eigenvalue of $D(f)$ with largest magnitude. A theorem of Fried implies that λ is also the eigenvalue with largest magnitude of the action of f_* on $H_1(M^\circ;\mathbb{Z})$, and also that λ occurs with multiplicity one [Fri85]. In particular, the eigenvalues of $\psi_0(f) = f_*|_{W_0}$ have modulus strictly less than $|\lambda|$. Again, $\psi_0(f)$ is not elliptic since ψ_0 is assumed to be faithful.

Now consider the quotient surfaces $S_1 = \mathbb{H}^2/D(F)$ and $S_2 = \mathbb{H}^2/\psi_0(F)$. For $i = 1,2$, let g_i denote the genus of S_i and let $n_i \geq 1$ denote the number of ends. We have $F = \pi_1(S_1) = \pi_1(S_2)$, so this induces a homotopy equivalence $\phi : S_1 \to S_2$. Thus, we have that $\mathrm{rk}\ F = 2g_i + n_i - 1$ for each i. By statement 1 above, we have $n_1 \leq n_2$. We will show that $g_1 = g_2$ and $n_1 = n_2$.

An element of the fundamental group of a surface is called *peripheral* if it is homotopic to a puncture. Assume that $n_1 < n_2$. Let $\gamma_1, \ldots, \gamma_{n_1} \in \pi_1(S_1)$ denote disjoint peripheral curves. Note that the homology classes of these curves are linearly dependent. Let $\gamma'_j =$

$\phi_*(\gamma_j) \in \pi_1(S_2)$. Note that since S_2 has $n_2 > n_1$ punctures, the homology classes of the curves $\gamma'_1, \ldots, \gamma'_{n_1}$ are linearly independent. This contradicts either the fact that ϕ is a homotopy equivalence, or that $n_1 < n_2$. Thus, $n_1 = n_2$.

By the previous two paragraphs, we may take the homotopy equivalence $\phi : S_1 \to S_2$ to be a homeomorphism. In addition, these surfaces have the same number of parabolic cusps. Thus $\psi_0(F)$ is a lattice in $SL(2, \mathbb{Z})$. For non-peripheral $\beta \in \pi_1(S_1)$ let $\ell_1(\beta)$ denote the length of the geodesic representative on S_1, and let $\ell_2(\beta)$ denote the length of the geodesic representative of $\phi_*(\beta)$. Theorem 3.1 of [Thu98] states that

$$\sup_{\beta \in \pi_1(S_1)} \frac{\ell_2(\beta)}{\ell_1(\beta)} \geq 1,$$

with equality only if $S_1 = S_2$. (This holds for any pair of complete, finite area, hyperbolic structures on the same surface.) This contradicts statement (2). □

The following immediate consequence illustrates the use of Theorem 22.

Corollary 23. *If M is any translation surface of genus 1 or 2 with non-elementary Veech group, then the Veech group of any recurrent $\mathbb{Z}$-cover has the same limit set. In particular, if M is a square tiled surface of genus 1 or 2 then the Veech group of any recurrent $\mathbb{Z}$-cover is Fuchsian of the first kind.*

6. Multi-twists

A *multi-twist* is an $f \in \mathit{Aff}(M^\circ)$ which preserves the cylinders in a cylinder decomposition and for which $D(f)$ is parabolic with eigenvalue 1. It is well known that if M is compact, and $D(f)$ is parabolic then some power of f is a multi-twist. The action of a multi-twist f on $H_1(M, P; \mathbb{Z})$ is given by the formula

$$f_* : x \mapsto x + \sum_j i(x, \gamma_j^\circ) t_j \gamma_j, \tag{1}$$

where j varies over the cylinders in the preserved decomposition. Here $\gamma_j \in H_1(M, P; \mathbb{Z})$ and $\gamma_j^\circ \in H_1(M^\circ; \mathbb{Z})$ denote the homology classes of the core curve in cylinder j (although the curves are the same they represent elements in different homology spaces, and we will use different notation to distinguish them). We denote by $\langle \gamma_j \rangle$ and $\langle \gamma_j^\circ \rangle$ the $\mathbb{Z}$-module spanned by these curves in their respective homology groups. The restriction of the action of f on cylinder j is a Dehn twist. The number $t_j \in \mathbb{Z}$ is the twist number of this Dehn twist. Each t_j is non-zero and they all have the same sign. If this sign is positive f is performing left Dehn twists and if it is negative f is performing right Dehn twists.

Let $\phi = f_* - I$. That is,

$$\phi : H_1(M, P; \mathbb{Z}) \to H_1(M, P; \mathbb{Z}), \quad x \mapsto \sum_j i(x, \gamma_j^\circ) t_j \gamma_j. \tag{2}$$

A direct application of Proposition 8 yields the following.

Proposition 24. *The multi-twist $f \in \mathit{Aff}(M^\circ)$ lifts to an $\widetilde{f} \in \mathit{Aff}(\widetilde{M}_w, M^\circ)$ if and only if $\phi(w) = 0$.*

A linear map g on a vector space V is called *unipotent of index n* if $(g - I)^n(V) = \mathbf{0}$.
Lemma 25.

(1) $f_* : H_1(M,P;\mathbb{Z}) \to H_1(M,P;\mathbb{Z})$ *is unipotent of index* 2. *In particular, ker* $\phi = Fix(f_*^k)$ *for all non-zero* $k \in \mathbb{Z}$.
(2) $\phi\big(H_1(M,P;\mathbb{Z})\big)$ *is a submodule of* $\langle \gamma_j \rangle$ *of full rank. Moreover, this rank is bounded from above by the genus of* M.
(3) *If* $D\big(Aff(M^\circ)\big)$ *is non-elementary, then both* $\mathbf{hol} \circ \phi\big(H_1(M,P;\mathbb{Z})\big)$ *and* $\mathbf{hol}(ker\ \phi)$ *are* $\mathbb{Z}$*-modules of rank* $[k:\mathbb{Q}]$*, where* k *is the holonomy field.*

Proof. We prove these statements in order. For all $x \in H_1(M,P;\mathbb{Z})$, $\phi(x)$ is a linear combination of the $\{\gamma_j\}$. But, $i(\gamma_i, \gamma_j^\circ) = 0$ for all i and j. This implies statement (1).

From equation (2), we infer that $\phi\big(H_1(M,P;\mathbb{Z})\big) \subset \langle \gamma_j \rangle$. Consider the map $\pi : H_1(M^\circ;\mathbb{Z}) \to H_1(M,P;\mathbb{Z})$ induced by the inclusion of $M^\circ \hookrightarrow M$. Define the map

$$\eta : H_1(M,P;\mathbb{Z}) \to \langle \gamma_j^\circ \rangle, \quad x \mapsto \sum_j i(x,\gamma_j^\circ) t_j \gamma_j^\circ.$$

Note that $\pi \circ \eta = \phi$. We claim that the image of η is a $\mathbb{Z}$-module of rank equal to rk $\langle \gamma_j^\circ \rangle$. If this is true, then the conclusion follows as $\pi\left(\langle \gamma_j^\circ \rangle\right) = \langle \gamma_j \rangle$. We now prove this claim. By non-degeneracy of $i : H_1(M,P;\mathbb{Z}) \times H_1(M^\circ;\mathbb{Z}) \to \mathbb{Z}$, it is equivalent to show that if $x \in ker(\eta)$ then $i(x,\gamma^\circ) = 0$ for all $\gamma^\circ \in \langle \gamma_j^\circ \rangle$. We will prove the contrapositive of this statement. Suppose $i(x,\gamma^\circ) \neq 0$ for some $\gamma^\circ \in \langle \gamma_j^\circ \rangle$. Then $i(x,\gamma_k^\circ) \neq 0$ for some k. We compute

$$i\big(x,\eta(x)\big) = i\big(x, \sum_j i(x,\gamma_j^\circ) t_j \gamma_j^\circ\big) = \sum_j t_j i(x,\gamma_j^\circ)^2.$$

Recall that each t_j is non-zero and has the same sign. In addition, $i(x,\gamma_k^\circ) \neq 0$, so $i\big(x,\phi(x)\big) \neq 0$. Therefore, $\eta(x) \neq 0$.

The inequality rk $\langle \gamma_j \rangle \leq genus(M)$ follows from topology. Note that the core curves of cylinders are disjoint. Cutting along $g+1$ closed curves on a surface of genus g necessarily disconnects the surface. Hence, the maximal rank of the span of the $\{\gamma_j\}$ is $genus(M)$, because the γ_j have disjoint representatives.

Now we will consider statement (3). Since $D\big(Aff(M^\circ)\big)$ is non-elementary we can conjugate $f \in Aff(M^\circ)$ to obtain a new $f' \in Aff(M^\circ)$ so that $D(f')$ has an eigenvector distinct from the eigenvector of $D(f)$. By applying an element of $SL(2,\mathbb{R})$ to M°, we may assume without loss of generality that the derivatives are of the form

$$D(f) = \begin{bmatrix} 1 & \sqrt{\mu} \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad D(f') = \begin{bmatrix} 1 & 0 \\ \pm\sqrt{\mu} & 1 \end{bmatrix},$$

for some $\mu > 0$. Then the surface M can be obtained from Thurston's construction of flat surfaces admitting pseudo-Anosov automorphisms [Thu88, §6]. In particular, the widths of all horizontal cylinders (resp. vertical cylinders) appear as the entries of an eigenvector of a Perron-Frobenius matrix with eigenvalue μ. So from the theory of such matrices, we know $\mu \in k$ and

$$\text{rk } \mathbf{hol}_x\big(H_1(M,P;\mathbb{Z})\big) = \text{rk } \mathbf{hol}_y\big(H_1(M,P;\mathbb{Z})\big) = \text{rk } \mathbb{Z}[\mu] = [k:\mathbb{Q}],$$

where $\mathbf{hol}_x(\gamma)$ and rk $\mathbf{hol}_y(\gamma)$ denote the x- and y-coordinates of the holonomy map respectively, and $\mathbb{Z}[\mu]$ is the $\mathbb{Z}$-module generated by μ. For all $\gamma \in H_1(M,P;\mathbb{Z})$, we have

$$\mathbf{hol} \circ \phi(\gamma) = D(f)\mathbf{hol}(\gamma) - \mathbf{hol}(\gamma) = \big(\mu \mathbf{hol}_y(\gamma), 0\big).$$

We conclude

$$\text{rk } \mathbf{hol} \circ \phi\big(H_1(M,P;\mathbb{Z})\big) = \text{rk } \mathbf{hol}_y\big(H_1(M,P;\mathbb{Z})\big) = [k:\mathbb{Q}].$$

On the other hand, $\gamma \in \mathit{ker}\ \phi$ if and only if $\mathbf{hol}_y(\gamma) = 0$. Thus

$$\text{rk } \mathbf{hol}(\mathit{ker}\ \phi) = \text{rk } \mathbf{hol}_x\big(H_1(M,P;\mathbb{Z})\big) = [k:\mathbb{Q}].$$

□

We first establish a corollary of statement (1) of the lemma.

Corollary 26. *Let $f \in \mathit{Aff}(M^\circ)$ be a multi-twist, and let $w \in W$. If $f_*(w) \neq w$, then $D\big(\mathit{Aff}(\widetilde{M}_w, M^\circ)\big)$ is infinite index in $D\big(\mathit{Aff}(M^\circ)\big)$.*

Recall the definition of the holonomy-free subspace W of $H_1(M,P;\mathbb{Z})$. Proposition 8 stated that an element $f \in \mathit{Aff}(M^\circ)$ lifted to an affine automorphism $\widetilde{f} \in \mathit{Aff}(\widetilde{M}_w, M)$ if and only if $f_*(w) = \pm w$.

Theorem 27 (Lifting multi-twists)**.** *Assume $f \in \mathit{Aff}(M^\circ)$ is a multi-twist and that $D\big(\mathit{Aff}(M^\circ)\big)$ is non-elementary. Let the notation be as above.*

$$\text{rk } W - \text{rk}(W \cap \mathit{ker}\ \phi) = \text{rk } \langle \gamma_j \rangle - [k:\mathbb{Q}] \leq g - [k:\mathbb{Q}]. \tag{3}$$

In particular, f_* acts trivially on W if and only if $\text{rk } \langle \gamma_j \rangle = [k:\mathbb{Q}]$.

Proof. By linearity of ϕ and statement (2) of Lemma 25,

$$\text{rk}(\mathit{ker}\ \phi) = \text{rk } H_1(M,P;\mathbb{Z}) - \text{rk } \phi(H_1(M,P;\mathbb{Z})) = \text{rk } W + 2[k:\mathbb{Q}] - \text{rk } \langle \gamma_j \rangle.$$

Now, note that $W \cap \mathit{ker}\ \phi = \mathit{ker}\ \mathbf{hol}|_{\mathit{ker}\ \phi}$. By linearity of $\mathbf{hol}$, we have

$$\text{rk}(\mathit{ker}\ \phi) = \text{rk}(W \cap \mathit{ker}\ \phi) + \text{rk } \mathbf{hol}(\mathit{ker}\ \phi) = \text{rk}(W \cap \mathit{ker}\ \phi) + [k:\mathbb{Q}],$$

with the last equality following from statement (3) of the lemma. Subtracting these two equations gives (3). The inequality follows from statement (2) of the lemma. □

As an illustration of the use of Theorem 27, we deduce:

Corollary 28. *Suppose M is square-tiled and has a cylinder decomposition in which all cylinders are homologous in $H_1(M,P;\mathbb{Z})$. Then the Veech group of any recurrent $\mathbb{Z}$-cover is Fuchsian of the first kind.*

Proof. In this case $\text{rk}\langle\gamma_j\rangle = 1$ and $k = \mathbb{Q}$, so $f_* \in \mathit{ker}\ \psi_0$. Since Df_* is nontrivial, the result follows from Theorem 21. □

Remark 29. *In* [HS08, Theorem 2]*, Hubert and Schmithüsen define a class of $\mathbb{Z}$-covers of square tiled surfaces $\mathcal{O}^\infty \to \mathcal{O}$. They show that if $\mathcal{O}$ has a one-cylinder decomposition, then the Veech group of $\mathcal{O}^\infty$ is Fuchsian of the first kind. Thus Corollary 28 is an extension of the results of* [HS09].

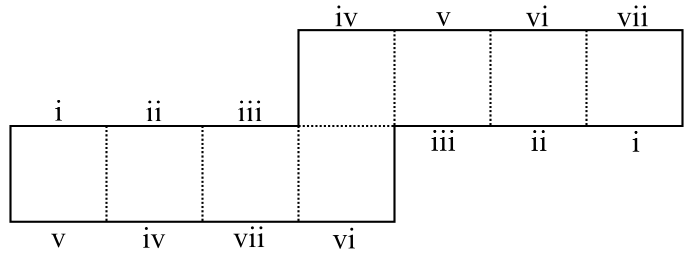

FIGURE 1. The eierlegende Wollmilchsau surface. Horizontal edges are glued together as indicated by the roman numerals. Vertical edges are glued to their opposite (by horizontal translations).

7. EXAMPLES

7.1. **Square tiled surfaces with homologous cylinders.** We give a construction of a square tiled surface with a horizontal cylinder decomposition all of whose cylinders are homologous. (In fact the reader may verify that all such surfaces arise via this construction.)

Let $C_0, \dots, C_{k-1}$ be cylinders all with the same rational circumference c, and each with rational width. For each $i = 0, \dots, k-1$ pick a rational interval exchange of $T_i : [0, c) \to [0, c)$. Use T_i to identify the bottom edge of C_i to the top edge of $C_{i+1 (\mathrm{mod}\ k)}$. Call the resulting surface M, and let $P \subset M$ be a finite set of points with rational coordinates. Then there is a horizontal cylinder decomposition of M°, all of whose cylinders are homologous. So, by Corollary 28, any recurrent $\mathbb{Z}$-cover of M° has a Veech group which is Fuchsian of the first kind.

The term *eierlegende Wollmilchsau* refers to the square tiled surface, W, whose properties were first studied by Herrlich and Schmithüsen [HS08]. It can be obtained by the above construction. See figure 1. This is a surface of genus three with four cone singularities, each with cone angle 4π. Let P denote the set of these singularities. The Veech group of W° is $\widehat{SL}(2, \mathbb{Z})$, the group of integer matrices of determinant ± 1.

Proposition 30. *Any recurrent $\mathbb{Z}$-cover of W° has a Veech group that contains the congruence 4 subgroup of $SL(2, \mathbb{Z})$.*

Proof. The horizontal direction has a multi-twist ϕ in a pair of homologous cylinders with derivative $D(\phi) = \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}$. For any $B \in \widehat{SL}(2, \mathbb{Z}) = \Gamma(W^\circ)$, there is a multi-twist ϕ_B in a pair of homologous cylinders with derivative $D(\phi_B) = BD(\phi)B^{-1}$. By Corollary 28, each ϕ_B lifts to any recurrent $\mathbb{Z}$-cover. The derivatives of these elements generate the congruence 4 subgroup of $SL(2, \mathbb{Z})$. □

7.2. **A question of Hubert and Schmithüsen.** We consider a surface defined in [HS09]. Let $Z_{3,1}$ be as in figure 2, let w be the cycle marked on figure 2 and let $Z_{3,1}^\infty$ be the corresponding $\mathbb{Z}$-cover. Since $\mathbf{hol}(w) = 0$ this is a recurrent $\mathbb{Z}$-cover. Hubert and Schmithüsen proved that the Veech group of $Z_{3,1}$ is not a lattice, but, since the genus of $Z_{3,1}$ is 2, rk $W_0 = 2$ so Theorem 22 implies that the Veech group of $Z_{3,1}^\infty$ is Fuchsian of the first kind. This answers a question raised in [HS09].

Since the Veech group of $Z_{3,1}$ is of the first kind but is not a lattice, it is infinitely generated. Note that a similar argument was employed in [HS04] and [McM03] to produce compact translation surfaces with infinitely generated Veech groups, and again in [HS09] to

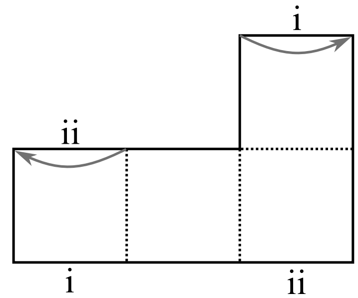

FIGURE 2. The surface $Z_{3,1}$ and the cycle w.

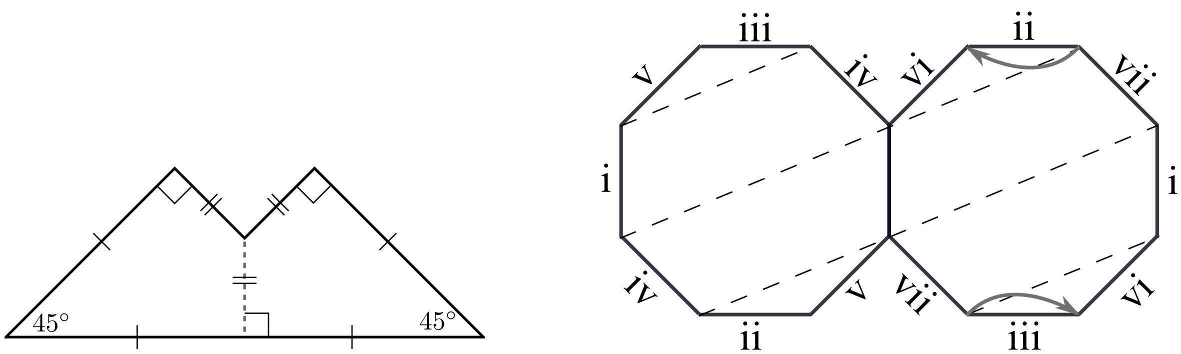

FIGURE 3. The polygon X and the surface O.

proved the existence of non-compact square-tiled surfaces with infinitely generated Veech group.

7.3. **A double cover of the octagon.** Let X denote the polygon shown on the left side of figure 3. The translation surface O is obtained by applying the Zemlyakov-Katok unfolding construction to X [ZK75]. The surface O is a double cover of the regular octagon with opposite sides identified, as depicted on the right side of figure 3. The surface O is of genus 3 with two cone singularities, each with cone angle 6π.

Let P consist of the two singularities of O. The orientation preserving part of the Veech group is generated by the derivatives of the following affine automorphisms.

- $h \in Aff(O^\circ)$ is the right multi-twist in the horizontal cylinder decomposition. We have $D(h) = \begin{bmatrix} 1 & 2+\sqrt{2} \\ 0 & 1 \end{bmatrix}$.
- $g \in Aff(O^\circ)$ is the right multi-twist in the cylinder decomposition in the direction of angle $\pi/4$. We have $D(g) = \begin{bmatrix} -\sqrt{2} & 1+\sqrt{2} \\ -1-\sqrt{2} & 2+\sqrt{2} \end{bmatrix}$.
- $f \in Aff(O^\circ)$ is the right multi-twist in the cylinder decomposition in the direction of angle $\pi/8$. $D(f) = \begin{bmatrix} -1-\sqrt{2} & 4+3\sqrt{2} \\ -\sqrt{2} & 3+\sqrt{2} \end{bmatrix}$.
- The two elements in $Aff(O^\circ)$ with derivative $-I$.

The orientation preserving part of the Veech group $D\big(Aff(O^\circ)\big)$ is an index two subgroup of a $(4, \infty, \infty)$-triangle group.

Proposition 31. *For any $w \in W \subset H_1(O,P;\mathbb{Z})$, there is a lift of $f \in Aff(O^\circ)$ to $D\big(Aff(\widetilde{O}_w, O^\circ)\big)$. In particular, $D\big(Aff(\widetilde{O}_w, O^\circ)\big)$ is always a Fuchsian group of the first kind.*

Proof. The affine automorphism f is a multi-twist which preserves a cylinder decomposition consisting of two cylinders. By the multi-twist theorem, it fixes all of W. By Theorem 21, $D\big(Aff(\widetilde{O}_w)\big)$ is a Fuchsian group of the first kind. □

The following gives an example of an infinite translation surface with non-arithmetic Veech group which is a lattice.

Proposition 32. *There exists a $w_1 \in W$ for which $D\big(Aff(\widetilde{O}_{w_1}, O^\circ)\big)$ is an infinitely generated Fuchsian group of the first kind, and a $w_2 \in W$ for which $D\big(Aff(\widetilde{O}_{w_2}, O^\circ)\big)$ contains the lattice $\langle D(f), D(g), D(h)\rangle \subset D\big(Aff(O^\circ)\big)$.*

Proof. We saw in the previous proposition that f always lifts. As O is genus 3, the multi-twist theorem implies that $Fix_W(g_*)$ and $Fix_W(h_*)$ are at worst codimension 1 inside W. Note that $dim\ W = 3$. Thus, we can find a non-zero $w_2 \in Fix_W(g_*) \cap Fix_W(h_*)$. As $D\big(Aff(O^\circ)\big)$ is generated by $\langle D(f), D(g), D(h)\rangle$, we see $D\big(Aff(\widetilde{O}_{w_2}, O^\circ)\big) = D\big(Aff(O^\circ)\big)$.

To see that there is a $w_1 \in W$ for which $D\big(Aff(\widetilde{O}_{w_1}, O^\circ)\big)$ is infinitely generated, it is sufficient to show that $D\big(Aff(\widetilde{O}_{w_2}, O^\circ)\big)$ is infinite index in $D\big(Aff(O^\circ)\big)$. By Corollary 26 and the multi-twist theorem, it is sufficient to check that the span of the core curves of a cylinder decomposition span a rank three submodule of $H_1(O,P;\mathbb{Z})$. This is true for both the horizontal direction and the direction of angle $\pi/4$. □

It turns out that there is only one non-zero $w \in W$ up to scaling which is fixed by f_*, g_* and h_*. This w is the homology class shown in grey in figure 3.

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