RESEARCH REPORT: A FAMILY OF RENORMALIZABLE POLYGON EXCHANGE MAPS

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Let X be a finite disjoint union of polygons in the plane. A *polygon exchange map* of X, $T : X \to X$, cuts X into finitely many pieces, then and applies a translation to each piece so that the image $T(X)$ has full area in X.

Polygon exchange maps are natural generalizations of interval exchange maps, and yet comparatively little is understood about polygon exchange maps. In particular, it is not understood how effective renormalization arguments will be for understanding the long-term behavior of iterating polygon exchange maps.

We will consider a family of rectangle exchange maps parameterized by a choice of a point (α, β) in the square $[0, \frac{1}{2}] \times [0, \frac{1}{2}]$. We denote these maps by $\Psi_{\alpha,\beta} : X \to X$, where X is a union of four tori. (These maps are defined at the end of this report.) We show that for irrational choices of α and β, there are points whose orbits under $\Psi_{\alpha,\beta}$ are periodic, with arbitrarily large period. The space X comes equipped with Lebesgue measure, λ, which we normalize so that $\lambda(X) = 1$. We define $M(\alpha, \beta)$ to be the λ-measure of the collection of all periodic points under $\Psi_{\alpha,\beta}$. In a forthcoming paper, we prove the following three results about this quantity.

Theorem 1 (Periodicity almost everywhere)**.** $M(\alpha, \beta) = 1$ *for Lebesgue-almost every parameter* $(\alpha, \beta) \in [0, \frac{1}{2}] \times [0, \frac{1}{2}]$.

As mentioned above, there are always periodic points. In fact, $M(\alpha, \beta) > 0$ for all (α, β). However,

Theorem 2 (Existence of periodicity only on small sets)**.** *For any* $\epsilon > 0$*, there are irrational parameters* α *and* β *so that* $M(\alpha, \beta) < \epsilon$.

Theorem 3 (Topologically generic aperiodicity)**.** *There is a dense set of irrational parameters* (α, β) *so that* $M(\alpha, \beta) \neq 1$.

Renormalization Dynamics

A *renormalization* of a polygon exchange map $T : X \to X$, is the choice of a finite union Y of sub-polygons of X with disjoint interiors such that the return map $T|_Y : Y \to Y$ is also a polygon exchange map. In the case of interval exchange maps, the return map to an interval is always another interval exchange map. For polygon exchange maps, however, not all such return maps yield polygon exchange maps.

Let G be the group of isometries of $\mathbb{R}$ generated by the maps $z \mapsto z + 1$ and $z \mapsto -z$. This group has $[0, \frac{1}{2}]$ as a fundamental domain, and for $x \in \mathbb{R}$ we write $x \pmod G$ to denote the unique element $y \in [0, \frac{1}{2}]$ so that there is a $g \in G$ with $g(x) = y$. We define the action f on

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the irrationals in $(0, \frac{1}{2})$ by

$$f(x) = \frac{x}{1-2x} \pmod{G}. \tag{1}$$

This map governs our renormalization.

For the maps $\Psi_{\alpha,\beta}$, we actually renormalize on a double cover. So, for each pair (α, β) we consider a lift $\widetilde{\Psi}_{\alpha,\beta} : \widetilde{X} \to \widetilde{X}$, where $\widetilde{X}$ is a particular double cover of X. For each pair of parameters, we show that there is a subset $Z = Z(\alpha, \beta) \subset \widetilde{X}$ so that the return map of $\widetilde{\Psi}_{\alpha,\beta}$ to Z is affinely conjugate to $\widetilde{\Psi}_{f(\alpha),f(\beta)}$.

So, we are implicitly interested in the dynamics of $f \times f$ on $[0, \frac{1}{2}] \times [0, \frac{1}{2}]$. The following results concern the dynamics of this map.

Proposition 4. *The measure ν on $[0, \frac{1}{2}]$ which is absolutely continuous with respect to λ with Radon-Nikodym derivative given by $\frac{d\nu}{d\lambda}(x) = \frac{1}{x} + \frac{1}{1-x}$ is f invariant.*

It should be noted that $\nu([0, \frac{1}{2}]) = \infty$. Nonetheless:

Theorem 5 (Poincaré recurrence holds)**.** *Let $A \subset [0, \frac{1}{2}] \times [0, \frac{1}{2}]$ be Lebesgue measurable. Then, for $\nu \times \nu$-a.e. pair $(\alpha, \beta) \in A$ there is an $n \geq 1$ so that $(f \times f)^n(\alpha, \beta) \in A$.*

It remains to explain what the orbit of (α, β) under $f \times f$ says about $M(\alpha, \beta)$. In fact, there is a formula which gives the quantity $M(\alpha, \beta)$ in terms of a limit involving a finite dimensional cocycle over $f \times f$. (We omit a formal description for brevity.) By explicitly working with this formula, we are able to show prove Theorem 3, as well as the following.

Lemma 6. *For α and β irrational, we have $1 - M(\alpha, \beta) < (1 - \alpha\beta)\Big(1 - M\big(f(\alpha), f(\beta)\big)\Big)$.*

Therefore, if the orbit of (α, β) has an accumulation point in $(0, \frac{1}{2}] \times (0, \frac{1}{2}]$, we have $M(\alpha, \beta) = 1$. In particular, Theorem 5 implies $M(\alpha, \beta) = 1$ for Lebesgue-almost every (α, β). (This proves Theorem 1.)

Definition of the polygon exchange maps $\Psi_{\alpha,\beta}$

We will now define the examples of interest to us. Consider the lattice $\Lambda = \mathbb{Z}^2 \cup \big[(\frac{1}{2}, \frac{1}{2}) + \mathbb{Z}^2\big]$, and let Y be the torus $\mathbb{R}^2/\Lambda$. This torus may be cut into two squares, $A_1 = [0, \frac{1}{2}) \times [0, \frac{1}{2})$ and $A_{-1} = [0, \frac{1}{2}) \times [\frac{1}{2}, 1)$, whose union forms a fundamental domain for the action of Λ on $\mathbb{R}^2$ by translation. Let N be the finite set of four elements, $N = \{(1, 0), (-1, 0), (0, 1), (0, -1)\}$. Fix two parameters $\alpha, \beta \in [0, \frac{1}{2}]$. We will define a polygon exchange map $\Psi_{\alpha,\beta} : Y \times N \to Y \times N$. Assume $(x, y) \in A_s$ and $\mathbf{v} = (a, b) \in N$. Then we define

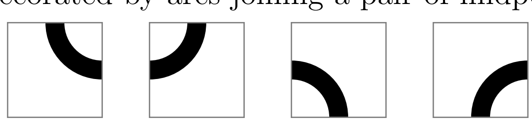
$$\Psi_{\alpha,\beta}\big((x, y), \mathbf{v}\big) = \big((x + bs\alpha, y + as\beta) \pmod{\Lambda}, (bs, as)\big). \tag{2}$$

Figure 1 illustrates this map.

Connections to Corner Percolation

We will very briefly describe the corner percolation model introduced by Bálint Tóth, and studied in depth by Gábor Pete [Pet08].

Consider four squares decorated by arcs joining a pair of midpoints of adjacent squares:



A *corner percolation tiling* is formed by tiling the plane with these tiles, so that whenever two tiles are adjacent along an edge, the arcs of these two tiles either both have endpoints on the edge or neither have endpoints along the edge. Thus, the arcs of the tiles join together to form a family of disjoint simple curves, which are either closed or bi-infinite.

In [Pet08], it was shown that in a "random" corner percolation tiling, all loops were closed. (Much stronger results were shown as well.) Corner percolation tilings can also be generated using symbolic dynamics applied to a pair of rotations by α and β. The quantity $M(\alpha, \beta)$ can then be interpreted as representing the probability that a curve of the tiling is closed, where the curve is chosen by fixing an edge in the tiling and looking a the curve through that edge. Thus, Theorems 1-3 are also theorems about tilings which are random with respect to a zero-entropy measure $\mu_{\alpha,\beta}$ on the space of corner percolation tilings.

References

[Pet08] Gábor Pete, *Corner percolation on $\mathbb{Z}^2$ and the square root of 17*, Ann. Probab. **36** (2008), no. 5, 1711–1747. MR 2440921 (2009f:60121)

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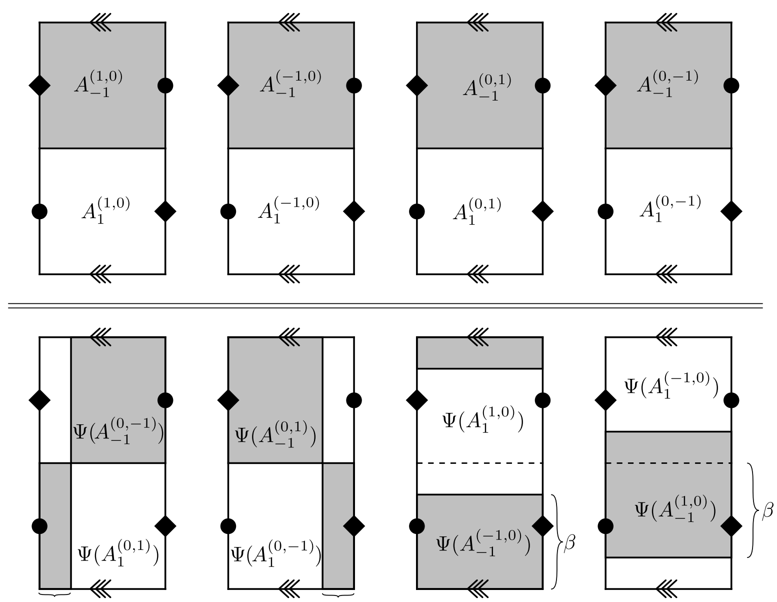

FIGURE 1. This illustrates the map $\Psi = \Psi_{\alpha,\beta}$ defined in equation 2. Above the line indicates the sets $A_s^{(a,b)} = A_s \times \{(a, b)\}$, and below illustrates their images under Ψ. In both cases, the tori are drawn $Y \times \{(1, 0)\}$, $Y \times \{(-1, 0)\}$, $Y \times \{(0, 1)\}$ and $Y \times \{(0, -1)\}$, from left to right.