GRID GRAPHS AND LATTICE SURFACES

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Abstract. First, we apply Thurston's construction of pseudo-Anosov homeomorphisms to grid graphs and obtain translation surfaces whose Veech groups are commensurable to (m, n, ∞) triangle groups. Many (if not all) of the surfaces we construct were first discovered by Bouw and Möller, however our treatment of the surfaces differs. We construct these surfaces by gluing together polygons in two ways. We use these elementary descriptions to compute the Veech groups, resolve primitivity questions, and describe the surfaces algebraically. Second, we show that some (m, n, ∞) triangle groups can not arise as Veech groups. This generalizes work of Hubert and Schmidt.

1. Introduction

A *translation surface* (X, ω) is a Riemann surface X equipped with a non-zero holomorphic 1-form ω. There is a well known action of $GL(2, \mathbb{R})$ on the moduli space of all translation surfaces. We use $GL(X, \omega) \subset GL(2, \mathbb{R})$ to denote the subgroup of all $A \in GL(2, \mathbb{R})$ for which $A(X, \omega) = (X, \omega)$. By area considerations, the determinant of an element $A \in GL(X, \omega)$ must be ± 1. The *Veech group* of (X, ω) is the group $SL(X, \omega) = GL(X, \omega) \cap SL(2, \mathbb{R})$. We define $PGL(X, \omega)$ and $PSL(X, \omega)$ to be the projections of these groups to $PGL(2, \mathbb{R})$ and $PSL(2, \mathbb{R})$, respectively. We say (X, ω) has the *lattice property* if the Veech group has finite co-volume in $SL(2, \mathbb{R})$.

Interest in these objects is sparked by connections with Teichmüller theory. See [MT02, §2.3], for instance. If (X, ω) is a translation surface, there is a totally geodesic isometric immersion of $\mathbb{H}^2/PSL(X, \omega)$ into $\mathcal{M}_g$, the moduli space of surfaces of genus $g = genus(X)$ equipped with the Teichmüller metric.

The (m, n, ∞) *triangle group* is the group

$$\langle a, b, c \ : \ a^2 = b^2 = c^2 = (ac)^m = (bc)^n = e \rangle.$$

This group can be realized as a subgroup $\Delta(m, n, \infty) \subset PGL(2, \mathbb{R}) = \mathrm{Isom}(\mathbb{H}^2)$ generated by reflections in the sides of a hyperbolic triangle with one ideal vertex and two angles of π/m and π/n. We use $\Delta^+(m, n, \infty)$ to refer to the orientation preserving part, $\Delta(m, n, \infty) \cap PSL(2, \mathbb{R})$.

We will describe translation surfaces (X, ω) for which $PGL(X, \omega)$ is conjugate to $\Delta(m, n, \infty)$ or an index two subgroup of $\Delta(m, n, \infty)$. In one sentence, these surfaces are constructed by applying Thurston's construction of pseudo-Anosov homeomorphisms to grid graphs. Sections 3 and 4 explain. We reprove the following theorem of Bouw and Möller [BM06].

Theorem 1 (Veech triangle groups). *Let m and n be integers satisfying $2 \leq m < n < \infty$.*

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- *If m and n are not both even, then there is a translation surface for which $PGL(X, \omega)$ is conjugate to $\Delta(m, n, \infty)$.*
- *When m and n are even, there is a translation surface for which $PGL(X, \omega)$ is conjugate to an index two subgroup of $\Delta(m, n, \infty)$. In this case, $\mathbb{H}^2/PSL(X, \omega)$ is a hyperbolic $(m/2, n/2, \infty, \infty)$-orbifold.*

A more detailed restatement of this theorem which also covers the case $m = n$ is provided by theorem 9. The surfaces we describe are the same as the surfaces constructed in [BM06] in many cases, and perhaps the same in the remaining cases. (See corollary 18 and question 19.) What sets this work apart is a concrete treatment of surfaces with these Veech groups. By applying Thurston's construction to grid graphs, we obtain a description of these surfaces by gluing together rectangles. We also provide a description of the surfaces in terms of a Riemann surface and a holomorphic 1-form in all cases. This was provided in [BM06] in the cases when m and n are relatively prime.

Our concrete treatment of these surfaces enabled us to discover finer information about these surfaces. See further below. Perhaps the following is more surprising.

Theorem 2 (Non-Veech triangle groups)**.** *Let m and n be even integers, and let $\gamma = \gcd(m, n)$. Under either the assumption that m/γ and n/γ are both odd or that $\gamma = 2$, there is no translation surface for which $\Delta^+(m, n, \infty) \subset PSL(X, \omega)$.*

This theorem contradicts a claim that appeared in a version of [BM06]. Hubert and Schmidt remarked a proof of this theorem in the special case when $m = 2$ [HS01, remark 7].

The following seems to be a very interesting open question.

Question 3. *For m and n even and not satisfying the conditions of theorem 2, is there a translation surface for which $\Delta^+(m, n, \infty) \subset PSL(X, \omega)$? Can the (m, n, ∞) orbifold be isometrically immersed in $\mathcal{M}_g$ for some g?*

We now return to a discussion of the translation surfaces constructed to prove theorem 1. New examples of surfaces with the lattice property can be constructed by carefully taking branched covers of existing examples. A translation surface is *primitive* if it does not arise as a branched covering of a translation surface of smaller genus. A discrete subgroup $\Gamma \subset SL(2, \mathbb{R})$ is *arithmetic* if Γ is conjugate to a finite index subgroup of $SL(2, \mathbb{Z})$. If $SL(X, \omega)$ is arithmetic, then (X, ω) is a cover of a torus, branched at one point. See theorem 5 and [GJ00].

As mentioned above, we will explicitly construct translation surfaces for which $PGL(X, \omega)$ is commensurable to $\Delta(m, n, \infty)$. We will explicitly compute the Veech group of these surfaces, and show that these surfaces are primitive whenever their Veech groups are non-arithmetic. There are only three arithmetic (m, n, ∞) triangle groups with $2 \leq m < n < \infty$. In [BM06], it was shown that these surfaces are primitive when m and n are relatively prime.

1.1. **Structure of paper.** We give some background on the problems described above in the next section. The major tool we use is the Thurston's construction of pseudo-Anosov homeomorphisms, which we describe in §3.

We give concrete descriptions of translation surfaces which meet the criteria of theorem 1 in §4. In §4.1, we apply the Thurston's construction to grid graphs to build these translation surfaces. Questions about topology, the Veech groups, arithmeticity, and primitivity are also answered in this section. In §4.2, we describe affinely equivalent surfaces built from "semiregular polygons." This description of these surfaces was independently discovered by

Ronen Mukamel. This point of view gives us algebraic formulas for the Riemann surface and holomorphic 1-form in each case. Sections 5 through 9 are devoted to proving these facts. We outline the structure of these sections in §4.3.

We prove theorem 2 in section 10.

2. Background

2.1. **Translation surfaces and the lattice property.** A *translation surface* (X, ω) is a Riemann surface X equipped with a non-zero holomorphic 1-form, ω. The 1-form ω provides local charts from X to $\mathbb{C}$ defined up to translation, away from the zeros of ω. At a zero, we have a chart to the Riemann surface $w = z^{k+1}$, where k is the order of the zero. A translation surface inherits a singular Euclidean metric by pulling back the Euclidean metric via the charts. From this point of view, a zero of order k of ω yields a cone singularity with cone angle $2\pi(k+1)$. In particular, we may equivalently think of a translation surface as a finite union of polygonal subsets of $\mathbb{C}$ with edges glued in pairs by translations. Cone singularities may appear at the equivalence class of a vertex of a polygon. Also, a translation surface inherits a notion of *direction* from $\mathbb{C}$. This is just the map which sends a tangent vector on X to its image vector in $\mathbb{C}$ under a chart. Note that the direction of a tangent vector is invariant under the geodesic flow in this metric.

Let (X, ω) and (X_0, ω_0) be translation surfaces. A homeomorphism $f : X \to X_0$ is called *affine* if it preserves the underlying affine structures. That is, there are real numbers a, b, c, d such that on local charts $f(x + iy) = (ax + by) + i(cx + dy)$. We use $f : (X, \omega) \to (X_0, \omega_0)$ to denote such a map. An *affine automorphism* of (X, ω) is an affine homeomorphism $f : (X, \omega) \to (X, \omega)$. Since f must preserve area we have $ad - bc = \pm 1$. The collection of all affine automorphisms forms the *affine automorphism group*, $\mathit{Aff}(X, \omega)$. The *derivative map* $D : \mathit{Aff}(X, \omega) \to GL(2, \mathbb{R})$ recovers the action of an affine automorphism on local charts. That is,

$$D : f \mapsto \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

We use $GL(X, \omega)$ to denote $D\big(\mathit{Aff}(X, \omega)\big) \subset GL(2, \mathbb{R})$. The orientation preserving part of $GL(X, \omega)$ is the *Veech group*, $SL(X, \omega) = D\big(\mathit{Aff}(X, \omega)\big) = SL(2, \mathbb{R}) \cap GL(X, \omega)$. As in the introduction, we use $PGL(X, \omega)$ and $PSL(X, \omega)$ to denote the projectivizations of these groups.

It is a theorem of Veech that $SL(X, \omega)$ is discrete and is never co-compact [Vee89]. We say that (X, ω) has the *lattice property* if $SL(X, \omega)$ has finite co-volume in $SL(2, \mathbb{R})$.

Interest in the lattice property arises from Teichmüller theory. Consider the hyperbolic plane, $\mathbb{H}^2 = SO(2, \mathbb{R}) \setminus SL(2, \mathbb{R})$. The quotient $\mathbb{H}^2/PSL(X, \omega)$ naturally immerses into the moduli space of complex structures on a surface with the genus of X. In fact, if we consider the related topic of *half-translation surfaces* (Riemann surfaces paired with a quadratic differential), every complete, finite area, totally geodesic subsurface of $\mathcal{M}_g$ is isometric to $\mathbb{H}^2/PSL(X, \omega)$, for some naturally related half-translation surface (X, ω) with the lattice property [MT02]. Veech found the first examples of translation surfaces with the lattice property [Vee89]. Since then, there has been interest in finding more examples and classifying these objects. See [KS00], [McM03], [Cal04], [BM06], and [McM06] for instance.

Let (X, ω) and (X_0, ω_0) be translation surfaces. A *covering* is a surjective (possibly branched) holomorphic map $f : X \to X_0$ for which the pullback form $f^*(\omega_0)$ equals ω.

(X, ω) is called *primitive* if it does not cover a surface of smaller genus. The following theorem of Möller answered a question of Hubert and Schmidt in [HS01]. See [Möl06, theorem 2.6] or [McM06, §2].

Theorem 4 (Möller)**.** *Every translation surface (X, ω) covers a primitive translation surface (X_0, ω_0). If the genus of X_0 is greater than 1, then this covering is unique and $SL(X, \omega)$ is a subgroup of $SL(X_0, \omega_0)$.*

We now concentrate on the special case when (X, ω) covers a torus (X_0, ω_0). We say two subgroups $\Gamma_1, \Gamma_2 \subset SL(2, \mathbb{R})$ are *commensurable* if are finite index subgroups $G_1 \subset \Gamma_1$ and $G_2 \subset \Gamma_2$ which are conjugate in $SL(2, \mathbb{R})$. A subgroup of $SL(2, \mathbb{R})$ is *arithmetic* if it is commensurable to $SL(2, \mathbb{Z})$. We have the following theorems about surfaces with arithmetic Veech groups.

Theorem 5 (Gutkin-Judge [GJ00])**.** *Let (X, ω) be a translation surface with the lattice property. The following are equivalent.*

(1) *(X, ω) covers a torus.*
(2) *$SL(X, \omega)$ is arithmetic.*

Theorem 6 (Schmithüsen [Sch04])**.** *If $SL(X, \omega)$ is arithmetic, then $SL(X, \omega)$ is conjugate to a subgroup of $SL(2, \mathbb{Z})$.*

Theorem 2 can be viewed a generalization of the following observation.

Corollary 7 (Some non-Veech triangle groups)**.** *The orientation preserving parts of the triangle groups $\Delta(2, 4, \infty)$, $\Delta(2, 6, \infty)$, $\Delta(4, 4, \infty)$ and $\Delta(6, 6, \infty)$ can not be subgroups of $PSL(X, \omega)$ for any translation surface (X, ω).*

Proof. The arithmetic triangle groups are classified in [Tak77], and include the listed triangle groups. These groups are not conjugate into $SL(2, \mathbb{Z})$ (or even $SL(2, \mathbb{Q})$), because they contain elements with irrational trace. □

3. Veech groups with non-commuting parabolics

We will introduce *Thurston's construction*, a combinatorial construction which produces a translation surface (X_0, ω_0) with hyperbolic elements in $SL(X, \omega)$. Thurston used this construction to generate pseudo-Anosov automorphisms of surfaces [Thu88, theorem 7]. It follows from work of Veech that all translation surfaces with the lattice property arise from this construction [Vee89, §9]. McMullen realized that a concise way to describe the combinatorics of this construction is via a bipartite ribbon graph [McM06, §4].

A *bipartite ribbon graph* is a finite connected graph $\mathcal{G}$ with vertex set $\mathcal{V}$ and edge set $\mathcal{E}$, equipped with two permutations $\mathfrak{n}, \mathfrak{e} : \mathcal{E} \to \mathcal{E}$, that satisfy the following conditions.

- The vertex set $\mathcal{V}$ is a disjoint union of two sets $\mathcal{A}$ and $\mathcal{B}$.
- There are functions $\alpha : \mathcal{E} \to \mathcal{A}$ and $\beta : \mathcal{E} \to \mathcal{B}$ such that every edge $e \in \mathcal{E}$ joins vertex $\alpha(e) \in \mathcal{A}$ to the vertex $\beta(e) \in \mathcal{B}$.
- For all $e \in \mathcal{E}$, the orbit $\mathcal{O}_{\mathfrak{e}}(e) = \{\mathfrak{e}^k(e) \ : \ k \in \mathbb{N}\}$ satisfies $\alpha(\mathcal{O}) = \alpha(e)$. Similarly, the orbit $\mathcal{O}_{\mathfrak{n}}(e) = \{\mathfrak{n}^k(e) \ : \ k \in \mathbb{N}\}$ satisfies $\beta(\mathcal{O}) = \beta(e)$.

The first two statements make $\mathcal{G}$ a bipartite graph. The third statement says that the cycles of the permutations $\mathfrak{e}$ and $\mathfrak{n}$ are the edges with a common vertex in $\mathcal{A}$ or a common vertex in $\mathcal{B}$, respectively. These cycles make $\mathcal{G}$ an oriented ribbon graph.

Given the data above plus a function $w : \mathcal{V} \to \mathbb{R}_{>0}$, we can construct a translation surface $(X_{\mathcal{G},w}, \omega_{\mathcal{G},w})$. This surface is a union of the rectangles R_e for $e \in \mathcal{E}$ with

$$R_e = [0, w \circ \beta(e)] \times [0, w \circ \alpha(e)].$$

To build $(X_{\mathcal{G},w}, \omega_{\mathcal{G},w})$ isometrically identify the right side of each rectangle R_e to the left side of $R_{\mathfrak{e}(e)}$ and the top side of R_e to the bottom side of $R_{\mathfrak{n}(e)}$. (This justifies the choice of symbols for the permutations; $\mathfrak{e}$ is for east and $\mathfrak{n}$ is for north.)

We now review some standard definitions. A *saddle connection* in a translation surface (X, ω) is a geodesic segment σ which intersects the zeros of ω precisely at its endpoints. A *cylinder* in (X, ω) is a closed subset isometric to a Euclidean cylinder of the form $[0, a] \times \mathbb{R}/k\mathbb{Z}$. The positive real constants a and k are the *width* and *circumference* of the cylinder, respectively. The interior of a cylinder in (X, ω) is foliated by periodic trajectories of the geodesic flow. The direction of each trajectory viewed as an element of $\mathbb{RP}^1$ is the same, and we call this the *direction* of the cylinder. Each boundary component of a cylinder is either a finite union of saddle connections or a periodic trajectory. A *cylinder decomposition* is finite collection of cylinders with disjoint interiors that cover the translation surface. Each cylinder in a decomposition has the same direction, so we call this the *direction of the cylinder decomposition*.

The surface $(X_{\mathcal{G},w}, \omega_{\mathcal{G},w})$ comes equipped with both a horizontal and a vertical cylinder decomposition. These horizontal and vertical cylinders are in bijective correspondence with the sets $\mathcal{A}$ and $\mathcal{B}$ respectively. Given $a \in \mathcal{A}$ and $b \in \mathcal{B}$, the respective cylinders are

$$\bigcup_{e \in \alpha^{-1}(a)} R_e \quad \text{and} \quad \bigcup_{e \in \beta^{-1}(b)} R_e.$$

We say that w is an *eigenfunction* of $\mathcal{G}$ corresponding to the *eigenvalue* $\lambda \in \mathbb{R}$ if for all $x \in \mathcal{V}$

$$\sum_{\overline{xy} \in \mathcal{E}} w(y) = \lambda w(x).$$

If w is a positive eigenfunction with eigenvalue λ, then the Veech group of $(X_{\mathcal{G},w}, \omega_{\mathcal{G},w})$ contains the elements

$$P_0 = \begin{bmatrix} 1 & \lambda \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad Q_0 = \begin{bmatrix} 1 & 0 \\ -\lambda & 1 \end{bmatrix}. \tag{1}$$

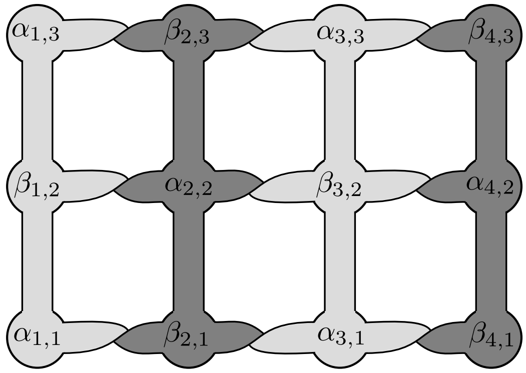
(See [Vee89, §9] and [McM06, §4], for instance.) Note, by the Perron-Frobeninus theorem, there is a unique positive eigenfunction up to scalar multiplication. For most applications, the choice of this eigenfunction is irrelevant. So, we will use $(X_{\mathcal{G}}, \omega_{\mathcal{G}})$ to denote $(X_{\mathcal{G},w}, \omega_{\mathcal{G},w})$ where w is a positive eigenfunction of the adjacency matrix of $\mathcal{G}$.

In §10, we will use the following consequence of comments in [Vee89, §9].

Theorem 8 (Veech)**.** *Given any (Y, η) such that $SL(Y, \eta)$ contains two non-commuting parabolics P and Q, there is a bipartite ribbon graph $\mathcal{G}$ and an affine homeomorphism $\phi : (Y, \eta) \to (X_{\mathcal{G}}, \omega_{\mathcal{G}})$. Moreover, we can assume that*

$$D(\phi) \circ P \circ D(\phi)^{-1} = \begin{bmatrix} \pm 1 & r\lambda \\ 0 & \pm 1 \end{bmatrix} \quad \textit{and} \quad D(\phi) \circ Q \circ D(\phi)^{-1} = \begin{bmatrix} \pm 1 & 0 \\ s\lambda & \pm 1 \end{bmatrix},$$

where λ is the Perron-Frobenius eigenvector of $\mathcal{G}$, $r, s \in \mathbb{Q}$ are non-zero, and the choice of signs depends on the sign of the eigenvalues of P and Q.



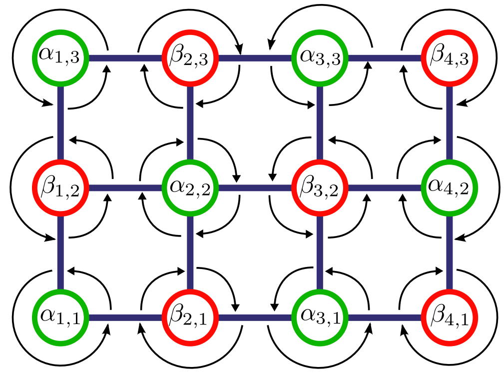


FIGURE 1. The bipartite ribbon graph $\mathcal{G}_{5,4}$ is shown on the left. On the right, we view this as a graph equipped with two edge permutations. The edge permutation $\mathfrak{e}$ is indicated by the arrows surrounding the α vertices, and $\mathfrak{n}$ is indicated by the arrows surrounding the β vertices.

4. VEECH TRIANGLE GROUPS

In this section, we describe our construction of translation surfaces $(X_{m,n}, \omega_{m,n})$ for which $PGL(X_{m,n}, \omega_{m,n})$ is commensurable to $\Delta(m, n, \infty)$. Section 4.3 reveals where we prove these results.

4.1. **Grid graphs.** For integers m and n with $m \geq 2$ and $n \geq 2$, we define the (m, n) *grid graph* to be the graph $\mathcal{G}_{m,n}$ whose vertices are $v_{i,j}$ for integers i and j satisfying $1 \leq i < m$ and $1 \leq j < n$. We define

$$\mathcal{E} = \{\overline{v_{i,j}v_{k,l}} \ : \ (i-k)^2 + (j-l)^2 = 1\}.$$

(Equivalently, embed the vertices in $\mathbb{R}^2$ in the natural way as in figure 1 and join edges between vertices of distance 1.) We make this graph bipartite by defining the disjoint subsets $\mathcal{A}, \mathcal{B}$ by

$$\mathcal{A} = \{v_{i,j} \in \mathcal{V} \ : \ i+j \text{ is even}\} \quad \text{and} \quad \mathcal{B} = \{v_{i,j} \in \mathcal{V} \ : \ i+j \text{ is odd}\} \tag{2}$$

We will use the notation $\alpha_{i,j} = v_{i,j}$ provided $v_{i,j} \in \mathcal{A}$ and $\beta_{i,j} = v_{i,j}$ provided $v_{i,j} \in \mathcal{B}$.

So that the edge set $\mathcal{E}$ is non-empty, we make the assumption that $mn \geq 6$. We make $\mathcal{G}_{m,n}$ a ribbon graph by defining the permutations $\mathfrak{e}, \mathfrak{n} : \mathcal{E} \to \mathcal{E}$ according to the following convention.

Convention 1 (Permutation convention)**.** The permutations $\mathfrak{e}, \mathfrak{n} : \mathcal{E} \to \mathcal{E}$ are determined from cyclic orderings for the edges around each vertex $v_{i,j}$. Consider the usual embedding of $\mathcal{G}_{m,n}$ into $\mathbb{R}^2$ as in figure 1. We choose the clockwise ordering around $v_{i,j}$ when i is even and the counter-clockwise ordering when i is odd.

The positive eigenfunction is given by the equation

$$w(v_{i,j}) = \sin(\frac{i\pi}{m}) \sin(\frac{j\pi}{n}). \tag{3}$$

For m and n as above, define the translation surface $(X_{m,n}, \omega_{m,n}) = (X_{\mathcal{G}_{m,n},w}, \omega_{\mathcal{G}_{m,n},w})$.

The following matrices are all 2×2 matrices of determinant -1 with eigenvalues ± 1.

$$A = \begin{bmatrix} -1 & -2\cos\frac{\pi}{m} \\ 0 & 1 \end{bmatrix} \quad B = \begin{bmatrix} -1 & 2\cos\frac{\pi}{n} \\ 0 & 1 \end{bmatrix} \quad C = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \quad E = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \tag{4}$$

The projection of the subgroup $\langle A, B, C\rangle \subset GL(2, \mathbb{R})$ to $PGL(2, \mathbb{R})$ is conjugate to the triangle group $\Delta(m, n, \infty)$ described in the introduction. The subgroup $\langle A, B, C\rangle \subset GL(2, \mathbb{R})$ is characterized by this image and the statement that $-I \in \langle A, B, C\rangle$. These matrices have the relations

$$A^2 = B^2 = C^2 = I \quad \text{and} \quad (AC)^m = (BC)^n = -I,$$

while AB is parabolic. When $n = m$, E appears as an additional symmetry, satisfying the relations $E^2 = I$, $EAE = B$ and $(EC)^2 = -I$. The subgroup $\langle A, C, E\rangle \subset GL(2, \mathbb{R})$ projects a group conjugate to $\Delta(2, m, \infty)$ in $PGL(2, \mathbb{R})$.

Theorem 9 (The Veech groups)**.** *Let $m \geq 2$ and $n \geq 2$ be integers with $mn \geq 6$.*

- *If $m \neq n$, then*
  - *When m and n are not both even, $GL(X_{m,n}, \omega_{m,n}) = \langle A, B, C\rangle$.*
  - *When m and n are both even, $GL(X_{m,n}, \omega_{m,n}) = \langle A, B, CAC, CBC\rangle$. (This is an index two subgroup of $\langle A, B, C\rangle$.)*
- *If $m = n$, then*
  - *When m is odd, $GL(X_{m,m}, \omega_{m,m}) = \langle A, C, E\rangle$.*
  - *When m is even, $GL(X_{m,m}, \omega_{m,m}) = \langle A, E, CAC\rangle$. (This is a reflection group in an $(\frac{m}{2}, \infty, \infty)$ triangle.)*

Corollary 10 (Arithmeticity)**.** *The surface $(X_{m,n}, \omega_{m,n})$ has an arithmetic Veech group if and only if (m, n) or (n, m) is in the set $\{(2, 3), (2, 4), (2, 6), (3, 3), (4, 4), (6, 6)\}$.*

Proof. It is straightforward to check that in all the listed cases the Veech group of $(X_{m,n}, \omega_{m,n})$ is conjugate to a subgroup of $SL(2, \mathbb{Z})$. We see that when (m, n) or (n, m) is $(4, 6)$ then $CACB$ is in the Veech group, but its trace is not an integer. In the remaining cases, the Veech group has elliptics whose trace is not rational. □

We will now consider primitivity. The graph $\mathcal{G}_{m,n}$ admits an automorphism ι defined by

$$\iota(v_{i,j}) = v_{m-i,n-j}. \tag{5}$$

In the special case that both m and n are even, this automorphism satisfies the following conditions.

- $\iota(\mathcal{A}) = \mathcal{A}$ and $\iota(\mathcal{B}) = \mathcal{B}$.
- $\iota \circ \mathfrak{e} = \mathfrak{e} \circ \iota$ and $\iota \circ \mathfrak{n} = \mathfrak{n} \circ \iota$.
- $w \circ \iota = w$.

These conditions imply that ι extends to an automorphism $\iota_* : (X_{m,n}, \omega_{m,n}) \to (X_{m,n}, \omega_{m,n})$, which simply permutes the rectangles making up $(X_{m,n}, \omega_{m,n})$ according to ι. In particular, $(X^e_{m,n}, \omega^e_{m,n}) = (X_{m,n}, \omega_{m,n})/\iota_*$ is a translation surface covered by $(X_{m,n}, \omega_{m,n})$.

Note that $SL(X^e_{m,n}, \omega^e_{m,n})$ is arithmetic if and only if $SL(X_{m,n}, \omega_{m,n})$ is.

Theorem 11 (Primitivity)**.** *When m and n are not both even, $(X_{m,n}, \omega_{m,n})$ is primitive unless $SL(X_{m,n}, \omega_{m,n})$ is arithmetic. When both m and n are even, $(X^e_{m,n}, \omega^e_{m,n})$ is primitive unless $SL(X^e_{m,n}, \omega^e_{m,n})$ is arithmetic.*

It remains to describe the Veech groups of $(X^e_{m,n}, \omega^e_{m,n})$.

Theorem 12. *Suppose m and n are even, and $(X^e_{m,n}, \omega^e_{m,n})$ is not a torus. Then, $GL(X^e_{m,n}, \omega^e_{m,n}) = GL(X_{m,n}, \omega_{m,n})$.*

Theorem 13 (The stratum of $(X_{m,n},\omega_{m,n})$)**.** *Let $m \geq 2$ and $n \geq 2$ and assume $mn \geq 6$. Set $\gamma = \gcd(m,n)$. Then $X_{m,n}$ is a surface of genus $\frac{mn-m-n-\gamma}{2}+1$. Provided $m>3$ or $n>3$, the 1-form $\omega_{m,n}$ has γ zeros, each of order $\frac{mn-m-n}{\gamma}-1$.*

Theorem 14 (The stratum of $(X^e_{m,n},\omega^e_{m,n})$)**.** *If both even integers $m \leq 4$ and $n \leq 4$, then $(X^e_{m,n},\omega^e_{m,n})$ is a torus. Now assume m and n are even and one is greater than four. Set $\gamma = \gcd(m,n)$. There are the following two cases.*

1. *If both m/γ and n/γ are odd, then genus$(X^e_{m,n}) = \frac{mn-m-n-2\gamma}{4}+1$, and $\omega^e_{m,n}$ has γ zeros each of order $\frac{mn-m-n}{2\gamma}-1$.*
2. *Otherwise, genus$(X^e_{m,n}) = \frac{mn-m-n-\gamma}{4}+1$, and the 1-form $\omega^e_{m,n}$ has $\gamma/2$ zeros each of order $\frac{mn-m-n}{\gamma}-1$.*

4.2. **Decomposition into semiregular polygons.** The (a,b)-*semiregular* $2n$-*gon* is the $2n$-gon whose edge vectors (oriented counterclockwise) are given by

$$\mathbf{v}_i = \begin{cases} a(\cos\frac{i\pi}{n},\sin\frac{i\pi}{n}) & \text{if } i \text{ is even} \\ b(\cos\frac{i\pi}{n},\sin\frac{i\pi}{n}) & \text{if } i \text{ is odd} \end{cases}$$

for $i = 0,\dots,2n-1$. Denote this $2n$-gon by $P_n(a,b)$. The edges whose edge vectors are $\mathbf{v}_i$ for i even are called *even edges*. The remaining edges are called *odd edges*. We restrict to the cases where $a \geq 0$ and $b \geq 0$, but $a \neq 0$ or $b \neq 0$. In the case where one of a or b is zero, $P_n(a,b)$ degenerates to a regular n-gon. In the case where $a=0$ or $b=0$ and $n=2$, $P_n(a,b)$ degenerates to an edge.

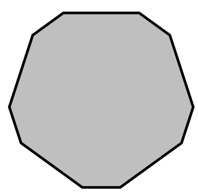


FIGURE 2. The semiregular polygon $P_5(1,2)$.

Fix m and n. Define the polygons $P(k)$ for $k = 0,\dots,m-1$ by

$$P(k) = \begin{cases} P_n(\sin\frac{(k+1)\pi}{m},\sin\frac{k\pi}{m}) & \text{if } n \text{ is odd} \\ P_n(\sin\frac{k\pi}{m},\sin\frac{(k+1)\pi}{m}) & \text{if } n \text{ is even and } k \text{ is even} \\ P_n(\sin\frac{(k+1)\pi}{m},\sin\frac{k\pi}{m}) & \text{if } n \text{ is even and } k \text{ is odd.} \end{cases} \tag{6}$$

We form a surface by identifying the edges of the polygons in pairs. For k odd, we identify the even sides of $P(k)$ with the opposite side of $P(k+1)$, and identify the odd sides of $P(k)$ with the opposite side of $P(k-1)$. The cases in the definition of $P(k)$ are chosen so that this gluing makes sense. We call the resulting surface $(Y_{m,n},\eta_{m,n})$. See examples in figures 3 and 4. Ronen Mukamel independently discovered these surfaces.

Theorem 15 (Semiregular decomposition)**.** *There is are affine homeomorphisms $\mu : (X_{m,n},\omega_{m,n}) \to (Y_{m,n},\eta_{m,n})$ and $\nu : (X_{m,n},\omega_{m,n}) \to (Y_{n,m},\eta_{n,m})$ with derivatives*

$$D(\mu) = \begin{bmatrix} \csc\frac{\pi}{n} & \cot\frac{\pi}{n} \\ 0 & -1 \end{bmatrix} \quad \textit{and} \quad D(\nu) = \begin{bmatrix} -\csc\frac{\pi}{m} & \cot\frac{\pi}{m} \\ 0 & -1 \end{bmatrix}.$$

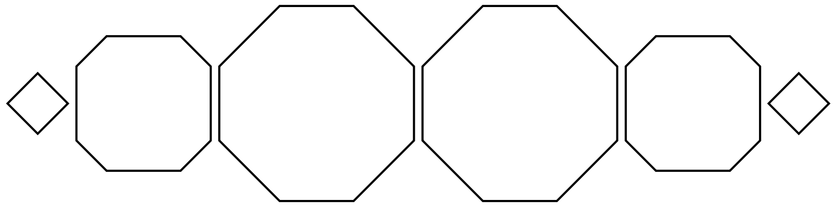


FIGURE 3. These polygons make up one component of the surface $(Y_{6,4}, \eta_{6,4})$. These are the polygons $P(0)$, $P(1)$, $P(2)$, $P(3)$, $P(4)$ and $P(5)$ from left to right.

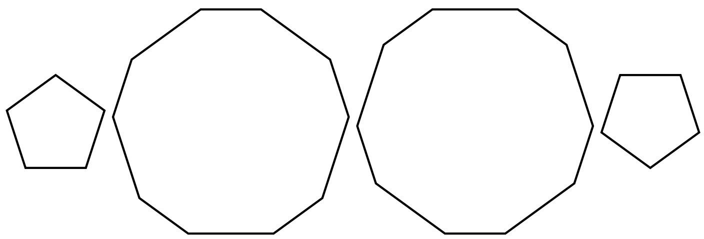


FIGURE 4. These polygons make up one component of the surface $(Y_{4,5}, \eta_{4,5})$. These are the polygons $P(0)$, $P(1)$, $P(2)$ and $P(3)$ from left to right.

Of course, when m and n are even, the surface $(Y_{m,n}, \eta_{m,n})$ is not primitive. In this case, the polygons $P(i)$ and $P(m-1-i)$ differ only by translation. In particular, there is an automorphism $\iota'_* : (Y_{m,n}, \eta_{m,n}) \to (Y_{m,n}, \eta_{m,n})$ which interchanges $P(i)$ and $P(m-1-i)$ for $i = 0, 1, \ldots, m/2-1$ and for which $D(\iota'_*) = I$. Moreover, $\iota'_* = \mu \circ \iota_* \circ \mu^{-1}$. (See proposition 22.) We use $(Y^e_{m,n}, \eta^e_{m,n})$ to denote $(Y_{m,n}, \eta_{m,n})/\iota'_*$. Given this, the following is a corollary the theorem above.

Corollary 16. *For m and n even, there are affine homeomorphisms $\mu^e : (X^e_{m,n}, \omega^e_{m,n}) \to (Y^e_{m,n}, \eta^e_{m,n})$ and $\nu^e : (X^e_{m,n}, \omega^e_{m,n}) \to (Y^e_{n,m}, \eta^e_{n,m})$, with $D(\mu^e) = D(\mu)$ and $D(\nu^e) = D(\nu)$.*

We will now give explicit formulas for the Riemann surfaces and 1-forms for the surfaces $(Y_{m,n}, \eta_{m,n})$ and $(Y^e_{m,n}, \eta^e_{m,n})$. Compare the following to [BM06, theorem 5.15]. (All but the third formula, which is closely connected to the second, appear in [BM06]. However, the formulas are applied more broadly here.)

Proposition 17. *Assume $mn \geq 6$. We have the following formulas for the primitive Riemann surfaces and holomorphic differentials $(Y_{m,n}, \eta_{m,n})$ and $(Y^e_{m,n}, \eta^e_{m,n})$ (up to scaling and rotating).*

(1) *If m is odd, then $Y_{m,n}$ is defined by $y^{2n} = (u-2) \prod_{j=1}^{(m-1)/2} \left(u - 2\cos\frac{2j\pi}{m}\right)^2$, and*

$$\eta_{m,n} = \frac{y\ du}{(u-2)\prod_{j=1}^{(m-1)/2}(u - 2\cos\frac{2j\pi}{m})}.$$

(2) *If m is even and n is odd, then $Y_{m,n}$ is defined by* $y^{2n} = (u-2)^n \prod_{j=1}^{m/2} \left(u - 2\cos\frac{(2j-1)\pi}{m}\right)^2$, *and* $\eta_{m,n} = \dfrac{y \ du}{(u-2)\prod_{j=1}^{m/2}(u - 2\cos\frac{(2j-1)\pi}{m})}$.

(3) *If both m and n are even, then $Y^e_{m,n}$ is defined by* $y^{n} = (u-2)^{\frac{n}{2}} \prod_{j=1}^{m/2} \left(u - 2\cos\frac{(2j-1)\pi}{m}\right)$, *and* $\eta^e_{m,n} = \dfrac{y \ du}{(u-2)\prod_{j=1}^{m/2}(u - 2\cos\frac{(2j-1)\pi}{m})}$.

Corollary 18. *When m and n are relatively prime, $(Y_{m,n}, \eta_{m,n})$ is the same as a surface constructed in* [BM06, theorem 5.15].

The following remains unresolved by this paper.

Question 19. *When* $\gcd(m, n) \neq 1$, *what is the relationship between the surfaces with Veech groups commensurable to $\Delta(m, n, \infty)$ constructed in this paper and constructed by Bouw and Möller* [BM06]*?*

4.3. **Locations of proofs.** The author has strived to make the proofs of each major result above readable independently. The paper has been separated into the following sections. The semiregular decomposition theorem is the main tool of the paper. We prove it in section 5. We study the topology of these surfaces in section 6. In section 7, we compute the Veech groups of these surfaces. We prove our primitivity results in section 8. Finally in section 9, we discuss our formulas for the Riemann surfaces and 1-forms given in proposition 17.

Remark 20. *In this version of the paper, nearly all results are proved in terms of the semiregular decomposition, and the grid graph description of the surfaces is just a bridge between the affinely equivalent surfaces $(Y_{m,n}, \eta_{m,n})$ and $(Y_{n,m}, \eta_{n,m})$. It is possible, though more cumbersome, to compute the Veech group and topology of these surfaces through the grid graph description. This was the point of view of earlier versions of this paper.*

5. The semiregular polygon decomposition

In this section we prove theorem 15, which provides a decomposition of the surface $(X_{m,n}, \omega_{m,n})$ into semiregular polygons, up to an affine transformation. The theorem provides two such decompositions. We will first prove the existence of $\mu : (X_{m,n}, \omega_{m,n}) \to (Y_{m,n}, \eta_{m,n})$, which provides a decomposition of $(X_{m,n}, \omega_{m,n})$ into semiregular $2n$-gons, up to an affine transformation. This is the difficult part of the theorem. Then, we will analyze the subgroup of $SL(2, \mathbb{R})$ which preserves the set of horizontal and vertical directions. This is a dihedral group of order 8. We will see that $(X_{m,n}, \omega_{m,n})$ and $(X_{n,m}, \omega_{n,m})$ differ only by an element of this dihedral group. In particular, the existence of μ will imply the existence of a $\nu : (X_{m,n}, \omega_{m,n}) \to (Y_{n,m}, \eta_{n,m})$.

5.1. **The existence of $\mu : (X_{m,n}, \omega_{m,n}) \to (Y_{m,n}, \eta_{m,n})$.** We begin by describing a decomposition of $(X_{m,n}, \omega_{m,n})$ into polygons. These will be the analogs of the polygons $P(0), \ldots, P(m-1)$ making up $(Y_{m,n}, \eta_{m,n})$.

For ease of exposition, we consider the augmented graph $\mathcal{G}'_{m,n}$ obtained by attaching *degenerate nodes* and *degenerate edges* to the graph $\mathcal{G}_{m,n}$. The nodes of $\mathcal{G}_{m,n}$ are in bijection with the coordinates $(i,j) \in \mathbb{Z}^2$ with $0 < i < m$ and $0 < j < n$. The nodes of $\mathcal{G}'_{m,n}$ will be in bijection with those $(i,j) \in \mathbb{Z}^2$ with $0 \le i \le m$ and $0 \le j \le n$. Our added nodes are called *degenerate nodes*. We join new *degenerate edges* between nodes of distance one in the plane that are not already joined by an edge. Our graph $\mathcal{G}'_{m,n}$ is also bipartite, and we follow the same naming conventions for nodes as when discussing $\mathcal{G}_{m,n}$. See equation 2, and the text below. An example graph is shown in figure 5.

Let $\mathcal{E}'$ denote the set of all edges of $\mathcal{G}'_{m,n}$. We call a degenerate edge $e \in \mathcal{E}'$ $\mathcal{A}$-*degenerate*, $\mathcal{B}$-*degenerate* or *completely degenerate* if ∂e contains a degenerate $\mathcal{A}$-node, a degenerate $\mathcal{B}$-node or both, respectively. We also define permutations $\mathfrak{e}', \mathfrak{n}' : \mathcal{E}' \to \mathcal{E}'$ following convention 1.

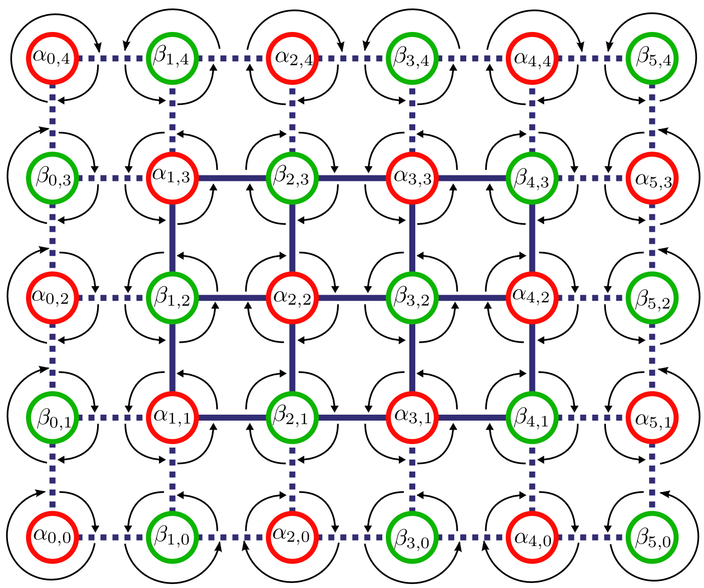


FIGURE 5. The augmented graph $\mathcal{G}'_{5,4}$. The degenerate edges are drawn as dotted lines. The map $\mathfrak{e}'$ is given by the arrows surrounding the α vertices, and the map $\mathfrak{n}'$ is given by the arrows surrounding the β vertices.

These degenerate edges correspond to degenerate rectangles on our surface $(X_{m,n}, \omega_{m,n})$. A *degenerate rectangle* is a rectangle with zero width or zero height. (The added nodes correspond to cylinders of zero width according to equation 3.) The $\mathcal{A}$-degenerate edges correspond to horizontal saddle connections (rectangles with zero height) and the $\mathcal{B}$-degenerate edges correspond to vertical saddle connections. The completely degenerate edges correspond to points on our surface.

Each edge $e \in \mathcal{E}'$ corresponds to a rectangle (or degenerate rectangle) $R_e = R(e)$ in the surface $(X_{m,n}, \omega_{m,n})$ with horizontal and vertical sides. The *positive diagonal* of a rectangle with horizontal and vertical sides is the diagonal with positive slope. For a degenerate rectangle, we take the positive diagonal to be the rectangle itself. Let $\mathbf{d}(e)$ denote the vector which points along the positive diagonal, oriented rightward and upward. The lower triangle, denoted $L(e)$, of a rectangle $R(e)$ is the triangle below the positive diagonal. The upper triangle, $U(e)$ is the triangle above the positive diagonal. See figure 6. For degenerate rectangles, we take $R(e) = L(e) = U(e)$ to be the corresponding saddle connection, or point.

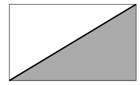


FIGURE 6. A rectangle's positive diagonal. The lower triangle is shaded gray, and the upper triangle is white.

Recall that $\mathcal{G}'_{m,n}$ is naturally embedded in $\mathbb{Z}^2$. We use $v_{i,j}$ to denote the node of $\mathcal{G}'_{m,n}$ in the position (i,j). We now define our decomposition of $(X_{m,n}, \omega_{m,n})$ into polygons. Let H_k denote the set of edges of $G'_{m,n}$,

$$H_k = \{\overline{v_{k,i}v_{k+1,i}} \in \mathcal{E}' \ : \ 0 < i < n\} \quad \text{for } k = 0, \ldots, m-1. \tag{7}$$

($\bigcup_k H_k$ is the set of horizontal edges in the graph $\mathcal{G}'_{m,n}$, and the edges in each H_k lie in a column.) For each such k define the polygon $Q(k) \subset (X_{m,n}, \omega_{m,n})$ by

$$Q(k) = \bigcup_{e \in H_k} R(e) \cup L\big(\mathfrak{n}'(e)\big) \cup L\big(\mathfrak{e}'^{-1}(e)\big) \cup U\big(\mathfrak{n}'^{-1}(e)\big) \cup U\big(\mathfrak{e}'(e)\big). \tag{8}$$

An example decomposition is shown in figure 7.

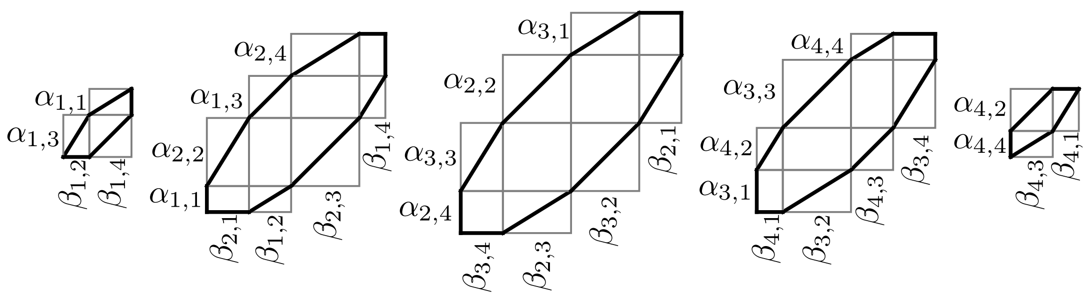


FIGURE 7. The surface $(X_{5,5}, \omega_{5,5})$ decomposes into the polygons $Q(0), Q(1), \ldots Q(4)$ ordered from left to right. Portions of the horizontal cylinders, α_*, and the vertical cylinders, β_* are labeled.

We have the following description of the affine homeomorphism $\mu : (X_{m,n}, \omega_{m,n}) \to (Y_{m,n}, \eta_{m,n})$. This implies half of theorem 15.

Lemma 21. *There is an affine homeomorphism $\mu : (X_{m,n}, \omega_{m,n}) \to (Y_{m,n}, \eta_{m,n})$ such that $\mu\big(Q(k)\big) = P(k)$ for $k = 0, \ldots, m-1$. Moreover,* $D(\mu) = \begin{bmatrix} \csc \frac{\pi}{n} & \cot \frac{\pi}{n} \\ 0 & -1 \end{bmatrix}$.

Proof. Let M denote the matrix identified as $D(\mu)$ in the lemma. The proof consists of two parts. First we show that $M\big(Q(k)\big)$ is the same as $P(k)$ up to translation. Second, we show that the boundary edges of the polygons $Q(k)$ considered as subsets of $(X_{m,n}, \omega_{m,n})$ are identified in the same combinatorial way as the polygons $P(k)$ which make up $(Y_{m,n}, \eta_{m,n})$. Concretely, we are defining the affine homeomorphism on pieces, and checking that the homeomorphisms agree on the boundaries. If this is true, then the homeomorphisms extend to the whole surface.

We will show that these subsets $Q(k)$ are in fact polygons, and $M\big(Q(k)\big) = P(k)$. We break into two cases depending on the parity of k.

Fix an odd integer k satisfying $0 < k \leq m-1$. Define the edge $e_i = \overline{v_{k,i}v_{k+1,i}}$ for $i = 0, \ldots, n$. We have $e_i \in H_k$ when $i = 1, \ldots, n-1$. We have that $\mathfrak{n}' \circ \mathfrak{e}'(e_i) = \mathfrak{e}' \circ \mathfrak{n}'(e_i) =$

e_{i+1}. Therefore many of the triangles are mentioned twice in equation 8 (e.g. $L\big(\mathfrak{n}'(e_1)\big) = L\big(\mathfrak{e}'^{-1}(e_2)\big)$.) Moreover, the top right coordinate vertex $R(e_i)$ is the same as the bottom left vertex of $R(e_{i+1})$ and this point is non-singular provided neither rectangle is degenerate. Thus this point is non-singular for $i = 1, \ldots, n-2$. With this in mind, we see that $Q(k)$ is formed by a chain of rectangles $R(e_i)$ moving to the northeast with some triangles added on. In particular, for $k = 1, \ldots, m-2$, $Q(k)$ has $2n$ sides. When $k = m-1$, half of these sides will degenerate to points. We treat these cases as a $2n$-gons as well, with half of their edges of length 0. Let $\mathbf{u}_i$ for $i = 0, \ldots, 2n-1$ be the edge vectors of $Q(k)$ oriented counterclockwise around $Q(k)$. We assume the first edge vector $\mathbf{u}_0$ is the lower horizontal edge of the rectangle $R(e_1)$. (We have $\mathbf{u}_0 = \mathbf{d}(\mathfrak{n}'^{-1}(e_1)) = \mathbf{d}(\overline{\alpha_{k+1,0}\beta_{k+1,1}})$.) We find these edge vectors of $Q(k)$ to be

$$\mathbf{u}_i = \begin{cases} \mathbf{d}(\overline{\alpha_{k+1,i}\beta_{k+1,i+1}}) & \text{if } i < n \text{ and } i \text{ even} \\ \mathbf{d}(\overline{\alpha_{k,i}\beta_{k,i+1}}) & \text{if } i < n \text{ and } i \text{ odd} \\ -\mathbf{d}(\overline{\beta_{k+1,2n-1-i}\alpha_{k+1,2n-i}}) & \text{if } i \geq n \text{ and } i \text{ even} \\ -\mathbf{d}(\overline{\beta_{k,2n-1-i}\alpha_{k,2n-i}}) & \text{if } i \geq n \text{ and } i \text{ odd.} \end{cases} \tag{9}$$

Therefore, we have

$$\mathbf{u}_i = \begin{cases} \sin\frac{(k+1)\pi}{m}(\sin\frac{(i+1)\pi}{n}, \sin\frac{i\pi}{n}) & \text{if } i \text{ is even} \\ \sin\frac{k\pi}{m}(\sin\frac{(i+1)\pi}{n}, \sin\frac{i\pi}{n}) & \text{if } i \text{ is odd.} \end{cases}$$

By a simple trigonometric calculation,

$$M\mathbf{u}_i = \begin{cases} \sin\frac{(k+1)\pi}{m}(\cos\frac{i\pi}{n}, \sin\frac{i\pi}{n}) & \text{if } i \text{ is even} \\ \sin\frac{k\pi}{m}(\cos\frac{i\pi}{n}, \sin\frac{i\pi}{n}) & \text{if } i \text{ is odd.} \end{cases}$$

Thus, $M(Q(k)) = P_n(\sin\frac{(k+1)\pi}{m}, \sin\frac{k\pi}{m})$, the same polygon as $P(k)$.

The case of k even with $0 \leq k \leq m-1$ is similar. Let $e_i = \overline{v_{k,i}v_{k+1,i}}$ for $i = 0, \ldots, n$. We have $\mathfrak{n}'^{-1} \circ \mathfrak{e}'^{-1}(e_i) = \mathfrak{e}'^{-1} \circ \mathfrak{n}'^{-1}(e_i) = e_{i+1}$. So, again the lower left and top right vertices are non-singular. But, the chain of rectangles $R(e_i)$ moves toward the southwest. Again, it can be observed that $Q(k)$ is a $2n$-gon, which is degenerate if $k = 0$ or $k = m-1$. We would like to compute the edge vectors $\mathbf{w}_i$ for $i = 0, \ldots, 2n-1$. We set $\mathbf{w}_0$ to be the lower horizontal edge of $Q(k)$. We see $\mathbf{w}_0 = \mathbf{d}\big(\mathfrak{e}(e_n)\big)$. Thus, we introduce the variable j defined by $i = n+j$. The edge vectors are defined as follows.

$$\mathbf{w}_i = \mathbf{w}_{n+j} = \begin{cases} -\mathbf{d}(\overline{\alpha_{k,j}\beta_{k,j+1}}) & \text{if } j \geq 0 \text{ and } j \text{ even} \\ -\mathbf{d}(\overline{\alpha_{k+1,j}\beta_{k+1,j+1}}) & \text{if } j \geq 0 \text{ and } j \text{ odd} \\ \mathbf{d}(\overline{\beta_{k,-j-1}\alpha_{k,-j}}) & \text{if } j < 0 \text{ and } j \text{ even} \\ \mathbf{d}(\overline{\beta_{k+1,-j-1}\alpha_{k+1,-j}}) & \text{if } j < 0 \text{ and } j \text{ odd.} \end{cases} \tag{10}$$

We see

$$\mathbf{w}_i = \mathbf{w}_{n+j} = \begin{cases} -\sin\frac{k\pi}{m}(\sin\frac{(j+1)\pi}{n}, \sin\frac{j\pi}{n}) & \text{if } j \text{ is even} \\ -\sin\frac{(k+1)\pi}{m}(\sin\frac{(j+1)\pi}{n}, \sin\frac{j\pi}{n}) & \text{if } j \text{ is odd.} \end{cases}$$

We have

$$M\mathbf{w}_i = M\mathbf{w}_{n+j} = \begin{cases} \sin\frac{k\pi}{m}(\cos\frac{i\pi}{n}, \sin\frac{i\pi}{n}) & \text{if } i-n=j \text{ is even} \\ \sin\frac{(k+1)\pi}{m}(\cos\frac{i\pi}{n}, \sin\frac{i\pi}{n}) & \text{if } i-n=j \text{ is odd.} \end{cases}$$

Thus, $M(Q(k)) = P_n(\sin \frac{k\pi}{m}, \sin \frac{(k+1)\pi}{m})$ when n is even and $M(Q(k)) = P_n(\sin \frac{(k+1)\pi}{m}, \sin \frac{k\pi}{m})$ when n is odd. In either case, we have $M(Q(k)) = P(k)$.

Finally, we note that the identification of edges of these polygons agrees with the gluing definition given in section 4.2. Fix k odd. We will see that the even sides of $Q(k)$ are identified with the opposite sides of $Q(k+1)$, and the odd sides of $Q(k)$ are identified with the opposite sides of $Q(k-1)$. This needs to be done in four cases, of which we will only do one.

Fix an even integer $i < n$, Then up to sign, $\mathbf{u}_i$ is the positive diagonal of the rectangle $R(\overline{\alpha_{k+1,j}\beta_{k+1,j+1}})$. The opposite side of the polygon $Q(k+1)$ is edge of $Q(k+1)$ with edge vector $\mathbf{w}_{i+n}$. This is the positive diagonal of the rectangle $R(\overline{\alpha_{k+1,j}\beta_{k+1,j+1}})$ oriented in the opposite direction. Thus, these two edges are identified in $(X_{m,n}, \omega_{m,n})$.

The remaining three cases to cover are when i is even and $i \geq n$, i is odd and $i < n$, and when i is odd and $i \geq n$. These cases are left to the motivated reader. $\square$

We have the following corollary about the action of the automorphism $\iota_* : (X_{m,n}, \omega_{m,n}) \to (X_{m,n}, \omega_{m,n})$, which exists when m and n are even. Recall, ι_* was induced by a graph automorphism $\iota : \mathcal{G}_{m,n} \to \mathcal{G}_{m,n}$. See equation 5 of §4.1.

Proposition 22. *Suppose m and n are even. Then $\iota_*\big(Q(k)\big) = Q(m-1-k)$ for $k = 0, \ldots m-1$. It follows that $\iota'_* = \mu \circ \iota_* \circ \mu^{-1} : (Y_{m,n}, \eta_{m,n}) \to (Y_{m,n}, \eta_{m,n})$ is an affine automorphism of $(Y_{m,n}, \eta_{m,n})$ with $D(\iota'_*) = I$ and $\iota'_*\big(P(k)\big) = P(m-1-k)$ for $k = 0, \ldots m-1$.*

Proof. Recall the definition of $H_k \subset \mathcal{E}'$ given in equation 7. Note that the graph automorphism ι extends naturally to $\mathcal{G}'_{m,n}$ and satisfies $\iota(H_k) = H_{m-1-k}$. By the definition of $Q(k)$ in equation 8, $\iota_*\big(Q(k)\big) = Q(m-1-k)$. $\square$

5.2. **The dihedral group.** The dihedral group of order 8, D_8, acts on the plane in a way that preserves the set of directions $\{\text{horizontal}, \text{vertical}\}$. In particular, if S is a translation surface with horizontal and vertical cylinder decompositions and if $M \in D_8$, then the natural affine homeomorphism $S \to M(S)$ preserves the collection of all horizontal and vertical cylinders. Thus, there is an action of D_8 on the data associated to the cylinder intersection graph. Note that the matrices C and E given in equation 4 generate D_8. We will record their actions on this data.

Proposition 23 (Action of D_8)**.** *Let $S[\mathcal{G}, (\mathcal{A}, \mathcal{B}), (\mathfrak{e}, \mathfrak{n}), w]$ denote the translation surface constructed from the bipartite ribbon graph $\mathcal{G}$ with vertex set $\mathcal{V} = \mathcal{A} \cup \mathcal{B}$, width function $w : \mathcal{V} \to \mathbb{R}_{>0}$, edge set $\mathcal{E}$, and edge permutations $\mathfrak{e}, \mathfrak{n} : \mathcal{E} \to \mathcal{E}$, as in §3. Then*

- *$C\big(S[\mathcal{G}, (\mathcal{A}, \mathcal{B}), (\mathfrak{e}, \mathfrak{n}), w]\big) = S[\mathcal{G}, (\mathcal{B}, \mathcal{A}), (\mathfrak{n}^{-1}, \mathfrak{e}^{-1}), w]$, and*
- *$E\big(S[\mathcal{G}, (\mathcal{A}, \mathcal{B}), (\mathfrak{e}, \mathfrak{n}), w]\big) = S[\mathcal{G}, (\mathcal{A}, \mathcal{B}), (\mathfrak{e}^{-1}, \mathfrak{n}), w]$.*

In our setting, this gives us the following.

Corollary 24. *There is an affine homeomorphism $\rho : (X_{m,n}, \omega_{m,n}) \to (X_{n,m}, \omega_{n,m})$ with derivative E. In particular, $E \in GL(X_{m,m}, \omega_{m,m})$ for all m.*

Proof. The graph homomorphism $\eta : \mathcal{G}_{m,n} \to \mathcal{G}_{n,m} : v_{i,j} \mapsto v_{j,i}$ satisfies $\eta(\mathcal{A}_{m,n}) = \mathcal{A}_{n,m}$, $\eta(\mathcal{B}_{m,n}) = \mathcal{B}_{n,m}$, $\eta \circ \mathfrak{e}_{m,n} \circ \eta^{-1} = \mathfrak{e}_{n,m}^{-1}$, $\eta \circ \mathfrak{n}_{m,n} \circ \eta^{-1} = \mathfrak{n}_{n,m}$, and $w_{m,n} \circ \eta = w_{n,m}$. $\square$

This corollary was the last piece we needed to prove the semiregular decomposition, theorem 15.

Proof of theorem 15. Lemma 21 handles the existence of μ, so we will concentrate on the existence of ν. Let μ' denote the affine homeomorphism $(X_{n,m}, \omega_{n,m}) \to (Y_{n,m}, \eta_{n,m})$ guaranteed by lemma 21. We define the affine homeomorphism $\nu : (X_{m,n}, \omega_{m,n}) \to (Y_{m,n}, \eta_{m,n})$ to be $\nu = \mu' \circ \rho$, with $\rho : (X_{m,n}, \omega_{m,n}) \to (X_{n,m}, \omega_{n,m})$ as in corollary 24. We have that

$$D(\nu) = D(\mu') \cdot D(\rho) = \begin{bmatrix} -\csc \frac{\pi}{m} & \cot \frac{\pi}{m} \\ 0 & -1 \end{bmatrix}.$$

□

6. Topological type

In this section, we compute the topological types of the surfaces $(Y_{m,n}, \eta_{m,n})$ and $(Y^e_{m,n}, \eta^e_{m,n})$. Recall $(Y_{m,n}, \eta_{m,n})$ decomposes into the semiregular polygons $P(0), \dots, P(m-1)$.

Proposition 25 (Singularities of $(Y_{m,n}, \eta_{m,n})$)**.** *Let $\gamma = \gcd(m,n)$. There are γ equivalence classes of vertices of the decomposition into polygons. Each of these points has cone angle $2\pi(mn - m - n)/\gamma$.*

Proof. Let $\mathbf{v}_0$ be a vector based at a vertex v of $P(0)$, which is pointing along the boundary of $P(0)$ in a counterclockwise direction. We will rotate this vector counterclockwise around the point V of $(Y_{m,n}, \eta_{m,n})$. $P(0)$ is a regular n-gon, so we reach $P(1)$ when we have rotated by $\pi - \frac{2\pi}{n}$. Inside $P(1)$ we may rotate by another $\pi - \frac{2\pi}{2n}$ until we reach $P(2)$, since $P(1)$ is a semiregular $2n$-gon. For $i = 1, \dots, m-2$, $P(i)$ is a regular $2n$-gon. Thus we repeat this process until we reach $P(m-1)$. Then $P(m-1)$ is a regular n-gon again, so we rotate by $\pi - \frac{2\pi}{n}$. Now the indices decrease. When we rotate by $(m-2)(\pi - \frac{2\pi}{2n})$ we reach $P(0)$. We have closed up if the total rotation we have done is a multiple of 2π. In general, we see that the cone angle at V is

$$x\big(2(\pi - \frac{2\pi}{n}) + 2(m-2)(\pi - \frac{2\pi}{2n})\big) = 2x\pi\frac{nm - n - m}{n},$$

where x is the smallest positive integer for which this number is a multiple of 2π. We see $x = n/\gamma$. This says the singularity at v has cone angle gives the $2\pi(mn - m - n)/\gamma$. Every singularity is of this form, and the sum of all the angles of polygons $P(0), \dots, P(m-1)$ is $2\pi(mn - m - n)$. Hence, there are γ total singularities. □

Corollary 26. *The Euler characteristic of $(Y_{m,n}, \eta_{m,n})$ is $m + n + \gamma - mn$.*

Proof. The decomposition $(Y_{m,n}, \eta_{m,n}) = \bigcup_{i=0}^{m-1} P(i)$ has m faces, $(m-1)n$ edges, and γ vertices. □

Proposition 27 (Singularities of $(Y^e_{m,n}, \eta^e_{m,n})$)**.** *If m/γ and n/γ are odd, then there are γ equivalence classes of vertices in the polygonal decomposition of $(Y^e_{m,n}, \eta^e_{m,n})$ and each has cone angle $\pi(mn - m - n)/\gamma$. Otherwise, when one of m/γ or n/γ is even, there are $\gamma/2$ equivalence classes of vertices and each has cone angle $2\pi(mn - m - n)/\gamma$.*

Proof. The proof proceeds in the same manner. Recall, $(Y^e_{m,n}, \eta^e_{m,n})$ is a double cover of $(Y_{m,n}, \eta_{m,n})$ with the polygons $P(0)$ and $P(m-1)$. As before, let $\mathbf{v}_0$ be a vector based at a vertex v of $P(0)$, which is pointing along the boundary of $P(0)$ in a counterclockwise direction. We rotate $\mathbf{v}_0$ counterclockwise. We continue rotating until we get back to $P(0) = P(m-1)$. At this point, we have rotated by $\pi\frac{nm-n-m}{n}$. The cone angle at this point will be

$$x\pi\frac{nm - n - m}{n},$$

where x is the smallest positive integer which makes this an integer multiple of 2π. Thus $x = n/\gcd(2n, m+n)$. We have

$$\gcd(2n, m+n) = \gamma \gcd(\frac{2n}{\gamma}, \frac{m}{\gamma} + \frac{n}{\gamma}) = \begin{cases} 2\gamma & \text{if both } \frac{m}{\gamma} \text{ and } \frac{n}{\gamma} \text{ are odd} \\ \gamma & \text{if one of } \frac{m}{\gamma} \text{ or } \frac{n}{\gamma} \text{ is even} \end{cases}$$

when both m/γ and n/γ are even. This determines the cone angle. The number of vertices follows by dividing the total angle by this cone angle. □

We can compute the Euler characteristic as before.

Corollary 28. *If both m/γ and n/γ are odd, then $\chi(X^e_{m,n}, \omega^e_{m,n}) = \frac{m+n+2\gamma-mn}{2}$. Otherwise, $\chi(X^e_{m,n}, \omega^e_{m,n}) = \frac{m+n+\gamma-mn}{2}$.*

7. The Veech groups

7.1. **The orthogonal groups of $(Y_{m,n}, \eta_{m,n})$.** Given a translation surface (X, ω), we define the *orthogonal group* $O(X, \omega) = GL(X, \omega) \cap O(2, \mathbb{R})$. These are the derivatives of affine automorphisms which preserve the Euclidean metric. We define the additional matrix

$$Y_n = \begin{bmatrix} \cos\frac{\pi}{n} & -\sin\frac{\pi}{n} \\ -\sin\frac{\pi}{n} & -\cos\frac{\pi}{n} \end{bmatrix} \tag{11}$$

The matrices E and Y_n generate a dihedral group of order $4n$, and satisfy the relations $E^2 = Y_n^2 = I$ and $(EY_n)^n = -I$.

Proposition 29 (Orthogonal group of $(Y_{m,n}, \eta_{m,n})$.)**.** *Suppose $(Y_{m,n}, \eta_{m,n})$ is not a torus. If m and n are not both even, then $O(Y_{m,n}, \eta_{m,n}) = \langle E, Y_n\rangle$. If both m and n are even, then $O(Y_{m,n}, \eta_{m,n}) = \langle E, Y_n E Y_n\rangle$, a dihedral group of order $2n$.*

Proof. Recall, $(Y_{m,n}, \eta_{m,n})$ is a union of the semiregular $2n$-gons $P(0), P(1), \ldots, P(m-1)$. Both E and $Y_n E Y_n$ are symmetries of every semiregular $2n$-gon. In particular, there are affine automorphisms of $(Y_{m,n}, \eta_{m,n})$ with derivatives E and $Y_n E Y_n$ which preserve the each of the polygons $P(0), P(1), \ldots, P(m-1)$. In addition, when m or n is odd, then $Y_n\big(P(i)\big) = P(m-1-i)$, up to translation. This action extends to an affine automorphism of $(X_{m,n}, \omega_{m,n})$ with derivative Y_n.

Conversely, suppose $M \in O(Y_{m,n}, \eta_{m,n})$. Then the associated affine automorphism must permute the shortest saddle connections. These are the boundaries of the polygons $P(0)$ and $P(m-1)$. (Since $(Y_{m,n}, \eta_{m,n})$ is not a torus, all the vertices are singularities by 25.) In particular, M must preserve the set of directions in which these shortest saddle connections point. When m and n are not both even, the group of matrices with this property is $\langle E, Y_n\rangle$. When both m and n are even, $\langle E, Y_n E Y_n\rangle$ is the group of matrices with this property. Consequently, M must be in this group. □

We also cover the case of $(Y^e_{m,n}, \eta^e_{m,n})$. The proof is nearly identical, so we omit it.

Proposition 30 (Orthogonal group of $(Y^e_{m,n}, \eta^e_{m,n})$.)**.** *Suppose m and n are even, and $(Y^e_{m,n}, \eta^e_{m,n})$ is not a torus. Then $O(Y_{m,n}, \eta_{m,n}) = \langle E, Y_n E Y_n\rangle$.*

The proof proceeds in the same manner.

7.2. **The Veech groups.** In this section, we prove theorems 9 and 12 which prescribe the Veech groups of the surfaces $(X_{m,n}, \omega_{m,n})$ and $(X^e_{m,n}, \omega^e_{m,n})$. The following establishes the Veech group of $(X_{m,n}, \omega_{m,n})$.

Proof of theorem 9. We ignore the case when $(X_{m,n}, \omega_{m,n})$ is a torus. These are the cases when $m \leq 3$ and $n \leq 3$. It can be checked that in these cases, $SL(X_{m,n}, \omega_{m,n})$ is conjugate to $SL(2, \mathbb{Z})$.

We the following relations between matrices.

$$A = D(\nu)^{-1} \circ E \circ D(\nu), \quad B = D(\mu)^{-1} \circ E \circ D(\mu) \quad \text{and}$$
$$C = -I \circ D(\mu)^{-1} \circ Y_n \circ D(\mu) = D(\nu)^{-1} \circ Y_m \circ D(\nu).$$

In particular, theorem 15 implies that $A, B \in GL(X_{m,n}, \omega_{m,n})$, by pulling back the automorphisms. Similarly, $C \in GL(X_{m,n}, \omega_{m,n})$ when m and n not both even, and $CAC, CBC \in GL(X_{m,n}, \omega_{m,n})$ when both m and n are even. When $m = n$, we have $E \in GL(X_{m,m}, \omega_{m,m})$ by corollary 24. For all m and n, this proves that the group described in theorem 9 is really contained in the $GL(X_{m,n}, \omega_{m,n})$.

Now we will see that this is the whole Veech group. Let $\Gamma_{m,n} \subset SL(2, \mathbb{R})$ denote the orientation preserving subgroup of the group described in theorem 9 as the Veech group of $(X_{m,n}, \omega_{m,n})$. We have shown above that $\Gamma_{m,n} \subset SL(X_{m,n}, \omega_{m,n})$.

Let M be an orbifold which is topologically a 2-sphere (possibly with punctures). Let Σ be the set of singularities of M. That is, Σ is the collection of cone points and punctures of M. In this specific case, the *Euler number* of M is given by the formula

$$\chi(M) = 2 + \sum_{\sigma \in \Sigma} (\frac{1}{|G_\sigma|} - 1), \tag{12}$$

where G_σ is the group associated to the singularity s, and $|G_\sigma|$ denotes the order of this group. Treat $1/|G_\sigma| = 0$ if G_σ is infinite, ie. when σ is a puncture. For more information on the Euler number of an orbifold see [Thu81, chapter 13]. Note that a hyperbolic orbifold must have negative Euler number. Moreover, if $M \to N$ is a covering map of degree d, then $\chi(N) = \chi(M)/d$. In particular, we have $\chi(M) \leq \chi(N)$ with equality implying that $M = N$. Note further that adding more singular points only lowers the Euler number.

We apply this argument to the case $M = \mathbb{H}^2/\Gamma_{m,n}$ and $N = \mathbb{H}^2/SL(X_{m,n}, \omega_{m,n})$. We know both M and N are spheres, because M is a sphere and covers N. We know that N has at least one puncture, corresponding to the horizontal cylinder decomposition. If both n and m are even, N must have another puncture corresponding to the vertical cylinder decomposition. Here, no element of $SL(X_{m,n}, \omega_{m,n})$ may send the horizontal direction to the vertical direction. This is because when m and n are even, the number of maximal horizontal and vertical cylinders of $(X_{m,n}, \omega_{m,n})$ differ by one.

Now we consider the finite order singularities. These are fixed points of maximal orthogonal subgroups (orthogonal up to conjugation). We utilize theorem 15 and proposition 29 to determine two subgroups of $SL(X_{m,n}, \omega_{m,n})$ which are orthogonal. For two subgroups to count in the formula 12, they must not differ by conjugation in $SL(X_{m,n}, \omega_{m,n})$. In particular, we use the fact that if they differ in orders, then they do not differ by conjugation. We have the following special cases of orthogonal subgroups, for which we can verify distinctness up to conjugation. Note that for the orbifold calculation, we consider the group order in $PSL(2, \mathbb{R})$, because $-I$ acts trivially on $\mathbb{H}^2$.

- If m and n are not both even and $m \neq n$, we have groups of order m and n.

- If m and n are even with $m \neq n$, we have groups of order $m/2$ and $n/2$.
- If $m = n$ is odd, we have at least one group of order m and one group of order a multiple of 2. (The group of order two is $\langle EC \rangle$. EC can not be conjugated to lie in the group of order m because 2 does not divide m.)
- If $m = n$ is even, we have at least one group of order $m/2$.

In all cases, we have determined that $\chi(M) \geq \chi(N)$ and thus $SL(X_{m,n}, \omega_{m,n}) = \Gamma_{m,n}$.

Finally, we consider orientation reversing elements. To see that $GL(X_{m,n}, \omega_{m,n})$ is as stated in the theorem, note that $A \in GL(X_{m,n}, \omega_{m,n})$. The orientation preserving subgroup is always index two inside a group with orientation reversing elements. Thus a single orientation reversing element plus the orientation preserving subgroup determine the whole group. □

The following establishes the Veech group of $(X^e_{m,n}, \omega^e_{m,n})$ for m and n even.

Proof of theorem 12. Again, by proposition 30 and corollary 16, we see $A, B, CAC, CBC \in GL(X^e_{m,n}, \omega^e_{m,n})$ by pulling back the actions of the dihedral groups. Thus $GL(X_{m,n}, \omega_{m,n}) \subset GL(X^e_{m,n}, \omega^e_{m,n})$.

Now we check that this is everything. To do this, note that there must be two cusps in $\mathbb{H}^2/SL(X^e_{m,n}, \omega^e_{m,n})$, because there are again a different number of horizontal and vertical cylinders. In addition for $m \neq n$, the maximal orthogonal subgroups (orthogonal up to conjugation) are of orders $m/2$ and $n/2$. So, $\mathbb{H}^2/SL(X^e_{m,n}, \omega^e_{m,n})$ has two cone points of these orders. When $m = n$, then we have at least one orthogonal group of order $m/2$. Then, an orbifold Euler number computation shows that it must be that $SL(X^e_{m,n}, \omega^e_{m,n}) = SL(X_{m,n}, \omega_{m,n})$. Again, we can see $GL(X^e_{m,n}, \omega^e_{m,n}) = GL(X_{m,n}, \omega_{m,n})$ by noting that A appears in both groups. □

8. Primitivity

To show primitivity, we consider the following consequence of the theorem 4 of Möller. Recall, if a translation surface has the lattice property, then it only covers a torus if its Veech group is arithmetic. Let (X, ω) and (X_0, ω_0) be translation surfaces and $f : X \to X_0$ be a covering. Following [HS01], we say f is a *balanced covering* if the image of every zero of ω is a zero of ω_0.

Proposition 31. *Let (X, ω) be a translation surface which does not cover a torus. Let $f : X \to X_0$ be the unique covering of a primitive translation surface (X_0, ω_0) guaranteed by theorem 4. If Aff(X, ω) acts transitively on the zeros of ω, then f is a* balanced covering.

Proof. We know X_0 is not a torus. Then ω_0 has a zero $z \in X_0$. Choose $x \in X$ so that $f(x) = z$. Such an x must be a zero of ω. Let $y \in X$ be another zero. Then there is a $\rho \in \mathit{Aff}(X, \omega)$ for which $\rho(x) = y$. Let $\phi : X_0 \to X_0$ be the affine automorphism with $D(\phi) = D(\rho)^{-1}$, which exists by theorem 4. This theorem also guarantees the uniqueness of the covering $X \to X_0$. Thus, $\phi \circ f \circ \rho = f$. In particular, $f(y) = \phi^{-1} \circ f(x) = \phi^{-1}(z)$ which must be a zero. □

In order to apply this, we need the following.

Proposition 32. *$\mathit{Aff}(X_{m,n}, \omega_{m,n})$ acts transitively on zeros of $\omega_{m,n}$. For m and n even, $\mathit{Aff}(X^e_{m,n}, \omega^e_{m,n})$ acts transitively on zeros of $\omega^e_{m,n}$.*

Proof. We use the semiregular decomposition given by theorem 15. Consider the surface $(Y_{m,n}, \eta_{m,n})$, which is a union of the semiregular $2n$-gons $P(0), P(1), \ldots, P(m-1)$. Let $v \in (Y_{m,n}, \eta_{m,n})$ be a vertex of $P(k)$. Then one of the adjacent edges to v joins $P(k)$ to $P(k-1)$. Thus, v is also a vertex of $P(k-1)$. By induction, we see that v is a vertex of $P(0)$. But, the group generated by a rotation by $\frac{2\pi}{n}$ preserves $P(0)$ setwise and acts transitively on the vertices of $P(0)$. This group action extends to a group of affine automorphisms of $(Y_{m,n}, \eta_{m,n})$. Note that essentially the same argument works for $(X^e_{m,n}, \omega^e_{m,n})$. □

We have the following proof of primitivity.

Proof of theorem 11. We only consider the case where m and n are not both even. A slight variant of the argument below also holds for $(X^e_{m,n}, \omega^e_{m,n})$.

Primitivity of $(Y_{m,n}, \eta_{m,n})$ is equivalent to primitivity of $(X_{m,n}, \omega_{m,n})$ by theorem 15.

Let $f : X'_{m,n} \to X_0$ be a covering of a primitive translation surface (X_0, ω_0). We know that X_0 is not a torus, since $(Y_{m,n}, \eta_{m,n})$ has the lattice property but $SL(Y_{m,n}, \eta_{m,n})$ is not arithmetic.

By the propositions above, f sends zeros of $\omega'_{m,n}$ to zeros of ω_0. Then it sends saddle connections to saddle connections. Thus, if P is a convex polygon in $(Y_{m,n}, \eta_{m,n})$ whose boundary edges are all saddle connections, then $f(P)$ also has this property in (X_0, ω_0). Suppose that f is generically k-to-one. Then given any such P, the collection of polygons in $f^{-1} \circ f(P)$ is a collection of k isometric polygons with disjoint interiors that are bounded by saddle connections and differ only by translation. Now consider the polygon $P(0) \subset X'_{m,n}$. The saddle connections bounding $P(0)$ are the shortest of all the saddle connections of $(Y_{m,n}, \eta_{m,n})$, and the only other saddle connections that are this short bound $P(m-1)$. But, $P(m-1)$ is not a translate of $P(0)$. So f is generically one-to-one, and $(Y_{m,n}, \eta_{m,n})$ is primitive. □

9. Formulas for the surfaces

In this section, we establish part 3 of proposition 17. The proofs of the remaining parts are almost identical. In addition, they are discussed in [BM06]. For this whole section, both m and n will be even.

The proofs are essentially an application of the Schwarz-Christoffel mapping. For background, see [DT02] for instance. This formula has been used in the past to find formulas for translation surfaces which arise from billiard tables [AI88] [War98].

For this whole section both m and n will be even. The surface $(Y^e_{m,n}, \eta^e_{m,n})$ admits several automorphisms whose derivatives are Euclidean reflections. Let D be a connected component of the complement of the union of fixed points of these automorphisms. Such a region is a simply connected polygonal subset of $(Y^e_{m,n}, \eta^e_{m,n})$ with $m/2+2$ vertices and edges. See figure 8 for an example.

Recall $(Y_{m,n}, \eta_{m,n})$ was made up of the semiregular $2n$-gons $P(0), \ldots, P(m-1)$. The automorphism $\iota'_* : (Y_{m,n}, \eta_{m,n}) \to (Y_{m,n}, \eta_{m,n})$ interchanges $P(k)$ with $P(m-1-k)$. We defined $(Y^e_{m,n}, \eta^e_{m,n}) = (Y_{m,n}, \eta_{m,n})/\iota'_*$. We may consider $(Y^e_{m,n}, \eta^e_{m,n})$ as a union of the polygons $P(0), \ldots, P(m/2-1)$ with the same identifications, except that half of the edges of $P(m/2-1)$ are identified in pairs.

Let $C(0), \ldots, C(m/2-1)$ denote the center points of the polygons $P(0), \ldots, P(m/2-1)$, respectively. There are pairs of edges of $P(m/2-1)$ which are identified in $(Y^e_{m,n}, \eta^e_{m,n})$. Let E denote one of these edges, and let M denote the midpoint of one of these edges. We can

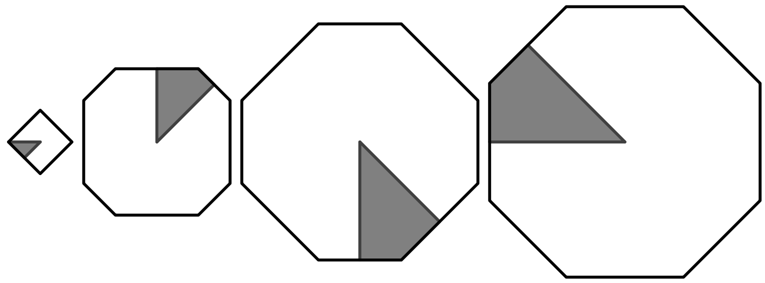


FIGURE 8. This is the surface $(Y^e_{8,4}, \eta^e_{8,4})$. The region bounded by the gray lines is a component of the complement of the collection of fixed points for affine automorphisms whose derivatives are Euclidean reflections.

describe a region D as follows. The path begins by traveling from a singularity along the edge E to the midpoint M. Then it makes a left turn of angle $\frac{\pi}{2}$. Then it travels to $C(m/2-1)$. Then it makes an left turn by angle $\frac{\pi}{n}$ and travels in a straight line to $C(m/2-2)$. Then it makes an angle of $\frac{\pi}{n}$ and travels along a line to $C(m/2-3)$. This process of rotating by $\frac{\pi}{n}$ and traveling to the center point of the polygon with one lower index repeats until we reach $C(1)$. From $C(1)$, it makes one last angle of $\frac{\pi}{n}$ and returns to the singularity while remaining inside $P(1)$.

We have determined the angles of D above. It remains to determine the lengths of some of its sides. Recall the definition of the semiregular $2n$-gon $P_n(a,b)$ given in §4.2. We define the *even radius* of $P_n(a,b)$ to be distance from the center to the midpoint of an even side (a side of length a). The *odd radius* is the distance from the center to the midpoint of an odd side. We have the following formula.

Proposition 33. *Let $r_n(a,b) = (b + a\cos\frac{\pi}{n})/(2\sin\frac{\pi}{n})$. Then, the even radius of $P_n(a,b)$ is $r_n(a,b)$ and the odd radius is $r_n(b,a)$.*

The proof is a simple analytic geometry problem. We have the following corollary.

Corollary 34. *Let $\kappa = (\cos\frac{\pi}{m} + \cos\frac{\pi}{n})/\sin\frac{\pi}{n}$. Then, the distance between $C(k-1)$ and $C(k)$ is $\kappa\sin\frac{k\pi}{m}$.*

Lemma 35. *There is a complex constant K so that the function*

$$f(u) = K\int_i^u (z-2)^{-\frac{1}{2}}\prod_{j=1}^{m/2}\left(z - 2\cos\frac{(2j-1)\pi}{m}\right)^{\frac{1}{n}-1}\,dz$$

induces a map from the upper half plane to $D \subset (Y^e_{m,n}, \eta^e_{m,n})$.

Given the lemma, the map extends to a holomorphic map from $X^e_{m,n}$, which is cover of the Riemann sphere branched only at points on the real axis, to the translation surface $(Y^e_{m,n}, \eta^e_{m,n})$ as described in §4.2. Moreover, this map is determined by integrating $\omega^e_{m,n}$, up to a complex scalar. Thus, this lemma directly implies part 3 of proposition 17.

Proof of lemma 35. This is the formula of a Schwarz-Christoffel mapping. The vertices of the image polygon will be the images of 2, $2\cos\frac{2j-1}{m}$ for $j = 1, \ldots, m/2$, and ∞. By the Schwarz-Christoffel mapping theorem, the angles at these points will be $\frac{\pi}{2}$, $\frac{\pi}{2n}$ for $j = 1, \ldots, m/2$, and $(\frac{m}{2} - \frac{m}{n} - \frac{1}{2})\pi$, respectively. These match the angles for D, if we send $2 \mapsto M$, $2\cos\frac{2j-1}{m} \mapsto$

$C(m/2-j)$ for $j=1,\ldots,m/2$, and ∞ to the singularity. To guarantee the lemma, it is sufficient to check the lengths of all but two of the edges are proportional to the lengths of the corresponding edges of D. (This is by the generalization of the "angle-side-angle" theorem taught in high school.) We will check the edges joining $C(m/2-j)$ to $C(m/2-j-1)$. The length of this edge is given by

$$\ell_j = \big|K \int_{2\cos(\frac{2j-1}{m}\pi)}^{2\cos(\frac{2j+1}{m}\pi)} (z-2)^{-\frac{1}{2}} \prod_{j=1}^{m/2} \Big(z - 2\cos\frac{(2j-1)\pi}{m}\Big)^{\frac{1}{n}-1}\, dz\big|.$$

Now, we make the change of coordinates $z = 2\cos(2w)$. After some trigonometric manipulations, we see

$$\ell_j = 2^{\frac{n+1}{n}}|K|\big|\int_{\frac{2j-1}{2m}\pi}^{\frac{2j+1}{2m}\pi} \frac{\cos w\ dw}{\cos(mw)^{1-\frac{1}{n}}}\big|.$$

Now make a change of coordinates $x = w - \frac{j\pi}{m}$. This yields

$$\ell_j = 2^{\frac{n+1}{n}}|K|\big|\int_{\frac{-\pi}{2m}}^{\frac{\pi}{2m}} \frac{\cos(x+\frac{j\pi}{m})\ dx}{\cos(mx)^{1-\frac{1}{n}}}\big| = 2^{\frac{n+1}{n}}|K|\big|\int_{\frac{-\pi}{2m}}^{\frac{\pi}{2m}} \frac{\cos(x)\cos(\frac{j\pi}{m}) - \sin(x)\sin(\frac{j\pi}{m})\ dx}{\cos(mx)^{1-\frac{1}{n}}}\big|$$

The sine term drops out by symmetry. Thus we have

$$\ell_j = 2^{\frac{n+1}{n}}|K|\cos(\frac{j\pi}{m})\int_{\frac{-\pi}{2m}}^{\frac{\pi}{2m}} \frac{\cos(x)\ dx}{\cos(mx)^{1-\frac{1}{n}}}$$

In particular ℓ_j is proportional to $\cos\frac{j\pi}{m}$. By the corollary, the distance between $C(m/2-j)$ to $C(m/2-j-1)$ should be proportional to

$$\sin\frac{(m/2-j)\pi}{m} = \cos\frac{j\pi}{m}.$$

So, ℓ_j is proportional to the distance from $C(m/2-j)$ to $C(m/2-j-1)$ as desired.

□

10. Lattices which are not Veech groups

The trace field of a subgroup $\Gamma \subset PSL(2,\mathbb{R})$ is the field $\mathbb{Q}(\text{tr }\Gamma) = \mathbb{Q}(\text{tr }\gamma\ :\ \gamma\in\Gamma)$. Note that the trace of an element of $PSL(2,\mathbb{R})$ is only defined up to sign, but these choices have no effect on the definition of this field. Let $\Gamma^{(2)} = \langle\gamma^2 \mid \gamma\in\Gamma\rangle$. This is a finite index subgroup of Γ. The *invariant trace field of* Γ is the field $\mathbb{Q}(\text{tr }\Gamma^{(2)})$. To abbreviate notation we use $k\Gamma$ to denote $\mathbb{Q}(\text{tr }\Gamma^{(2)})$. The name is justified by the fact that if Γ is finitely generated non-elementary group, then $k\Gamma' = k\Gamma$ for any finite index subgroup $\Gamma' \subset \Gamma$. In particular, $k\Gamma \subset \mathbb{Q}(\text{tr }\Gamma') \subset \mathbb{Q}(\text{tr }\Gamma)$. See [MR03, §3] for further background on this subject.

Lemma 36. *Suppose that $\Gamma = PSL(X,\omega)$ is finitely generated and non-elementary. Then $k\Gamma = \mathbb{Q}(\text{tr }\Gamma)$.*

This lemma is follows from results of Kenyon and Smillie [KS00]. Hubert and Schmidt realized that the following is implied by theorem 28 of [KS00].

Theorem 37 (Kenyon-Smillie)**.** *Let (X,ω) be a translation surface. If $A \in PSL(X,\omega)$ is hyperbolic, then $PSL(X,\omega)$ is conjugate into $PSL\big(2,\mathbb{Q}(\text{tr }A)\big)$.*

This theorem was used in [HS01, remark 7] to show that $\Delta^+(2, m, \infty)$ can not arise as $PSL(X, \omega)$ when m is even. The weaker statement of lemma 36 is all that is necessary for our proof that certain triangle groups can not arise as a $PSL(X, \omega)$, and the full strength of this theorem of Kenyon and Smillie does not exclude any additional triangle groups. We have chosen the proof using lemma 36 because it yields a more conceptually natural proof.

Proof of Lemma 36. Let $B \in \Gamma = PSL(X, \omega)$ be hyperbolic. B^2 is also hyperbolic. Thus,

$$\mathbb{Q}(\text{tr } B^2) \subset k\Gamma \subset \mathbb{Q}(\text{tr } \Gamma) \subset \mathbb{Q}(\text{tr } B^2),$$

with the last containment following from applying theorem 37 with $A = B^2$. $\square$

Remark 38 (A second proof of lemma 36)**.** *Another method of proving lemma 36 would be to generalize work of Gutkin and Judge* [GJ00]*. Their work implies that if $\Gamma = PSL(X, \omega)$ contains a finite index subgroup $\Gamma' \subset \Gamma$ which is conjugate into $PSL(2, \mathbb{Q})$ then Γ must be conjugate into $PSL(2, \mathbb{Q})$ as well. Their proof works with $\mathbb{Q}$ replaced by any subfield of $\mathbb{R}$.*

With regard to triangle groups, we have the following.

Lemma 39. *Suppose $2 \leq m \leq n < \infty$, $n > 2$, and let $\Gamma_{m,n} = \Delta^+(m, n, \infty) \subset PSL(2, \mathbb{R})$. Then $k\Gamma_{m,n} = \mathbb{Q}(\text{tr } \Gamma_{m,n})$ unless one of the following statements holds.*

(1) $\gcd(m, n) = 2$.
(2) *Both m and n are even, and both $m/\gcd(m, n)$ and $n/\gcd(m, n)$ are odd.*

Given this lemma, the proof of theorem 2 of the introduction follows by concatenating lemmas 36 and 39. The remainder of this section will be devoted to proving lemma 39. We will heavily use the book of Maclachlan and Reid [MR03].

The group $\Gamma_{m,n} = \Delta^+(m, n, \infty) \subset PSL(2, \mathbb{R})$ is generated by the projections of the following matrices to $PSL(2, \mathbb{R})$.

$$X = \begin{bmatrix} 0 & -1 \\ 1 & 2\cos\frac{\pi}{m} \end{bmatrix} \quad Y = \begin{bmatrix} -2\cos\frac{\pi}{n} & 1 \\ -1 & 0 \end{bmatrix} \tag{13}$$

These matrices satisfy the identities $X^m = Y^n = -I$ (which projects to the identity in $PSL(2, \mathbb{R})$, while XY is parabolic.

The following follows from lemma 3.5.3 of [MR03].

Proposition 40 (Trace field)**.** $\mathbb{Q}(\text{tr } \Gamma_{m,n}) = \mathbb{Q}(\cos\frac{\pi}{m}, \cos\frac{\pi}{n})$.

Note that if $m = 2$, then $\mathbb{Q}(\text{tr } \Gamma_{m,n}) = \mathbb{Q}(\cos\frac{\pi}{n})$. The following follows from lemmas 3.5.7 and 3.5.8 of [MR03].

Proposition 41 (Invariant trace field)**.** $k\Gamma_{m,n} = \mathbb{Q}(\cos\frac{2\pi}{m}, \cos\frac{2\pi}{n}, \cos\frac{\pi}{m}\cos\frac{\pi}{n})$.

Note that in the special case that $m = 2$, we have $k\Gamma_{m,n} = \mathbb{Q}(\cos\frac{2\pi}{n})$.

Proposition 42. $[\mathbb{Q}(\text{tr } \Gamma_{m,n}) : k\Gamma_{m,n}] \leq 2$. *Let $p, q \in k\Gamma_{m,n}[x]$ denote the polynomials*

$$p(x) = 2x^2 + (\cos\frac{2\pi}{m} - 1) \quad \textit{and} \quad q(x) = 2x^2 + (\cos\frac{2\pi}{n} - 1).$$

(1) *In the case $m = 2$, $\mathbb{Q}(\text{tr } \Gamma_{m,n})$ is the splitting field of q.*
(2) *Otherwise, $\mathbb{Q}(\text{tr } \Gamma_{m,n})$ is both the splitting field for p and the splitting field for q.*

Proof. By the double angle formula, $\cos\frac{\pi}{m}$ and $\cos\frac{\pi}{n}$ are roots of p and q, respectively. Combined with the knowledge that $\mathbb{Q}(\text{tr }\Gamma_{2,n}) = \mathbb{Q}(\cos\frac{\pi}{n})$ and $k\Gamma_{2,n} = \mathbb{Q}(\cos\frac{2\pi}{n})$, this implies statement (1). Note that $\cos\pi/m\cos\pi/n \in k\Gamma_{m,n}$. Thus when $m \neq 2$, $\cos\frac{\pi}{m} \in k\Gamma_{m,n}$ implies $\cos\frac{\pi}{n} \in k\Gamma_{m,n}$, and vice versa. This implies statement (2). □

We break the remainder of the proof of lemma 39 into special cases.

Corollary 43 (The case $m = 2$)**.** *$k\Gamma_{2,n} = \mathbb{Q}(\text{tr }\Gamma_{2,n})$ if and only if n is odd.*

Proof. Let $\zeta_n = e^{i\pi/n}$. We have that $[\mathbb{Q}(\zeta_n^2) : k\Gamma_{2,n}] = [\mathbb{Q}(\zeta_n) : \mathbb{Q}(\text{tr }\Gamma_{2,n})] = 2$, as our fields of interest are the real subfields of cyclotomic fields. We note that

$$[\mathbb{Q}(\zeta_n^2) : \mathbb{Q}] = \varphi(n) \quad \text{and} \quad [\mathbb{Q}(\zeta_n) : \mathbb{Q}] = \varphi(2n),$$

where φ denotes the Euler φ function. From the standard formula for φ, we have that

$$[\mathbb{Q}(\text{tr }\Gamma_{2,n}) : k\Gamma_{2,n}] = \frac{\varphi(2n)}{\varphi(n)} = \begin{cases} 1 & \text{if } n \text{ is odd} \\ 2 & \text{if } n \text{ is even.} \end{cases}$$

□

Corollary 44 (The odd case)**.** *If $m > 2$ and either m or n is odd, then $k\Gamma_{m,n} = \mathbb{Q}(\text{tr }\Gamma_{m,n})$.*

Proof. If m is odd then the polynomial p splits over $\mathbb{Q}(\cos 2\pi/m)$ and hence over $k\Gamma_{m,n}$. (A proof equivalent to that of corollary 43 applies.) Similarly, when n is odd, q splits over $\mathbb{Q}(\cos 2\pi/n)$. □

The following finishes the proof of lemma 39.

Proposition 45 (The even case)**.** *Assume $m > 2$ and both m and n are even. Then $k\Gamma_{m,n} = \mathbb{Q}(\text{tr }\Gamma_{m,n})$ unless $\gcd(m,n) = 2$ or both $m/\gcd(m,n)$ and $n/\gcd(m,n)$ are odd.*

Proof. Let $x = mn/\gamma$ and $\zeta = e^{\frac{2\pi i}{2x}}$, so ζ is a $2x$-th root of unity. We consider the cyclotomic field $\mathbb{Q}(\zeta)$. Note that

$$2\cos\frac{\pi}{m} = \zeta^{x/m} + \zeta^{-x/m} \quad \text{and} \quad 2\cos\frac{\pi}{n} = \zeta^{x/n} + \zeta^{-x/n}.$$

The Galois automorphisms of $\mathbb{Q}(\zeta)$ over $\mathbb{Q}$ are all induced by $\zeta \mapsto \zeta^k$ where k is an integer with $\gcd(k, 2x) = 1$ and $1 \le k < 2x$. We denote this Galois automorphism by σ_k.

The field $\mathbb{Q}(\zeta)$ contains both $k\Gamma_{m,n}$ and $\mathbb{Q}(\text{tr }\Gamma_{m,n})$. Assume that $k\Gamma_{m,n} \neq \mathbb{Q}(\text{tr }\Gamma_{m,n})$. Then, we know $[\mathbb{Q}(\text{tr }\Gamma_{m,n}) : k\Gamma_{m,n}] = 2$. Hence, there is a unique non-trivial Galois automorphism $\sigma \in Aut_{k\Gamma_{m,n}}\mathbb{Q}(\text{tr }\Gamma_{m,n})$ [Hun74, corollary V.4.3]. By the fundamental theorem of Galois theory, this σ is the restriction of a Galois automorphism of $\mathbb{Q}(\zeta)$ over $\mathbb{Q}$. In particular, it must be that $\sigma = \sigma_k|_{[\mathbb{Q}(\text{tr }\Gamma_{m,n})}$ for some k with $\gcd(k, 2x) = 1$ and $1 \le k < 2x$. Such an automorphism must be an involution satisfying

$$\sigma_k(\cos\frac{\pi}{m}) = -\cos\frac{\pi}{m} \quad \text{and} \quad \sigma_k(\cos\frac{\pi}{n}) = -\cos\frac{\pi}{n}, \tag{14}$$

as it must act transitively on the roots of both p and q. Note that any σ_k satisfying equation 14 fixes all elements of $k\Gamma_{m,n}$, but acts non-trivially on elements of $\mathbb{Q}(\text{tr }\Gamma_{m,n})$. In particular, $k\Gamma_{m,n} \neq \mathbb{Q}(\text{tr }\Gamma_{m,n})$ if and only if there is a Galois automorphism satisfying equation 14.

Consider the set if all k for which $\sigma_k(2\cos\frac{\pi}{m}) = -2\cos\frac{\pi}{m}$. This is equivalent to saying that $\sigma_k(\zeta^{x/m}) = \zeta^{x \pm x/m}$. In particular, this implies that

$$k \equiv \frac{x \pm x/m + 2ax}{x/m} \pmod{2x}$$

for some integer a. Similarly, $\sigma_k(2\cos\frac{\pi}{n}) = -2\cos\frac{\pi}{n}$ implies

$$k \equiv \frac{x \pm x/n + 2bx}{x/n} \pmod{2x}$$

for some b. By simplifying, we see this is equivalent to the conditions that

$$k \equiv m \pm 1 + 2am \pmod{2x} \quad \text{and} \quad k \equiv n \pm 1 + 2bn \pmod{2x}.$$

The existence of such a k is equivalent to the statement that there are choices of $a, b \in \mathbb{Z}$ and $\epsilon \in \{-2, 0, 2\}$ for which

$$m - n + \epsilon \equiv 2bn - 2am \pmod{2x}.$$

Since $\gamma = \gcd(m, n)$, the right hand side can be any even multiple of γ. Therefore, this is equivalent to the statement that

$$m - n + \epsilon \equiv 0 \pmod{2\gamma}. \tag{15}$$

We will now check to see when equation 15 holds for various choices of ϵ. First assume $\epsilon = 0$, then everything is divisible by γ, so this equation is equivalent to

$$m/\gamma - n/\gamma \equiv 0 \pmod{2}.$$

This equation is true if and only if both m/γ and n/γ are odd.

Now assume $\epsilon = \pm 2$. Notice that $m - n$ is always a multiple of γ. In particular, either

$$m - n \equiv 0 \pmod{2\gamma} \quad \text{or} \quad m - n \equiv \gamma \pmod{2\gamma}.$$

As $\epsilon = \pm 2$, the only possible values of $m - n + \epsilon$ are ± 2 or $\gamma \pm 2$ modulo 2γ. In particular for equation 15 to be true, we must have $\gamma = 2$. In this case, there is always a choice of $\epsilon \in \{-2, 0, 2\}$ which makes the equation true. We choose $\epsilon = 0$ if both m/γ and n/γ are odd, and $\epsilon = \pm 2$ otherwise. (The choice of sign is irrelevant in this case.) □

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