THE BOUW-MÖLLER LATTICE SURFACES AND EIGENVECTORS OF GRID GRAPHS

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ABSTRACT. This article investigates a family of translation surfaces whose Veech groups are lattices closely related to triangle groups. Many of these examples were discovered by Bouw and Möller through non-elementary means, and lack a simple elementary description. This article describes many of the Bouw-Möller examples by gluing together polygons in simple ways, and discovers some closely related new examples. We connect these examples to the principal eigenvectors of grid graphs, and provide analogous constructions of infinite genus translation surfaces with the lattice property.

A *Teichmüller curve* is a totally geodesic embedding of a complete two-dimensional hyperbolic orbifold into the moduli space of surfaces genus g equipped with the Teichmüller metric. Veech found the first examples of these objects [Vee89], and found Teichmüller curves in each genus $g \geq 2$. Since then, there has been interest in finding more examples and classifying these objects. (See [KS00], [McM03], [Cal04] and [McM06] for instance.) Teichmüller curves are naturally associated to flat structures and translation surfaces with the lattice property. See section 1.1.

Relatively recently, Bouw and Möller found Teichmüller curves isometric to $\mathbb{H}^2/\Gamma$ for Γ a hyperbolic (m, n, ∞) triangle group with $m < n < \infty$ [BM06]. Their examples generalize the lists found by Veech [Vee89] and Ward [War98].

The Bouw-Möller examples were found using techniques from algebraic geometry and Hodge theory. The purpose of this article is to provide a more elementary description of some of the Bouw-Möller examples as flat structures built by gluing together polygons. We achieve this goal in two different ways. We first construct the surfaces the surfaces as a union of rectangles. We give a second description of these surfaces as a union of *semiregular polygons*, $2n$-gons invariant under the dihedral group of order $2n$, $D_{2n} \subset \mathrm{Isom}(\mathbb{R}^2)$.

We only provide a description of the Bouw-Möller examples corresponding to m and n not both even. The case of m and n both even which appears in [BM06] is not treated here. However, for m and n both even, we do provide related examples of Teichmüller curves isometric to $\mathbb{H}^2/\Gamma$ with Γ index two inside the (m, n, ∞) triangle group. These examples appear to be new as we note in corollary 18.

As a secondary purpose, we describe an infinite list of Teichmüller curves for infinite genus surfaces. This generalizes the example found in [Hoo07] and also the example found by Hubert and Weiss [HW08].

Section 1 states our results after providing the necessary background. In section 1.2, we define the *cylinder intersection graph*, which keeps track of combinatorial data coming from a decomposition of a translation surface into rectangles. In section 1.3, we provide a description of these surfaces in terms of the cylinder intersection graph, which turns out

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to be grid graphs. *Grid graphs* are rectangular subgraphs of the usual square tiling of the plane. We provide our second description of these surfaces in terms of semiregular polygons in section 1.4.

In section 2, we prove that these finite surfaces have the lattice property. In section 3, we see why the geometry of these flat structures must correspond to the principal eigenvector of the cylinder intersection graph. We use this viewpoint to construct some infinite genus surfaces with the lattice property in section 3.2. We have devoted section 6 to a discussion of analogs of the decomposition into semiregular polygons which hold in these infinite genus cases.

The remainder of the paper is devoted to the more tedious proofs. In section 4, we prove that the translation surfaces described by a decomposition into rectangles and the surfaces described by a decomposition into semiregular polygons are the same up to the action of an element of the affine group. In section 5, we show that when n and m are not both even that our surfaces are the same as the surfaces of Bouw and Möller.

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1. Background and statement of results

1.1. **Background.** We consider a *polygon* to be an equivalence class of polygonal subsets of $\mathbb{R}^2$, where two polygonal subsets are equivalent if they differ by a translation or a rotation by π. A polygon P inherits the notion of direction from $\mathbb{R}^2$. Our notion of direction is a fibration $dir : T_1P \to \mathbb{R}/\pi\mathbb{Z}$. The map θ measures Euclidean angle modulo π of a vector compared to the horizontal.

A *flat structure* is union of polygons with all edges identified in pairs either by translations or rotations by π. Thus a flat structure is a Euclidean cone surface, all of whose cone angles are multiples of π, and whose holonomy group is contained in the group of translations and rotations by π of the plane. See [MT02] for more details on flat structures. A *flat structure* is called a *translation surface* when all of its cone angles are multiples of 2π, and when the holonomy group is contained in the group of translations.

Typically, flat structures and translation surfaces are taken to be closed surfaces (compact surfaces without boundary). However, we will also be interested in the infinite genus case. Our infinite genus surfaces will be non-compact and without boundary.

There is a natural action of $GL(2,\mathbb{R})/\pm I$ on the space translation surfaces and flat structures. Simply consider a flat structure as a union of polygons, $S = \bigcup_{i\in\Lambda} P_i$, and consider $A \in GL(2,\mathbb{R})/\pm I$ to be acting affinely on the plane. Then $A(S) = \bigcup_{i\in\Lambda} A(P_i)$ with edges identified in the same way as in S. (Note that $-I$ acts trivially on polygons in the sense defined above, which is why we consider the action of $GL(2,\mathbb{R})/\pm I$.) We abuse notation by using A both for the element $A \in GL(2,\mathbb{R})/\pm I$ and as the natural map $A : S \to A(S)$ defined above. The image of $\{A(S) \mid A \in PSL(2,\mathbb{R})\}$ in Teichmüller space under uniformization is known as a Teichmüller disk of S, and is totally geodesic in the Teichmüller metric.

Given a flat structure S, we say that $A \in GL(2,\mathbb{R})/\pm I$ is in the *Veech group of* S, $A \in V(S)$, if there is a direction preserving isometry $\iota_A : A(S) \to S$. The composition $\iota_A \circ A : S \to S$ is known as an *affine automorphism of* S, and the set of all such compositions is known as the *affine automorphism group of* S, *Aff*(S). We say the derivative $D(\iota_A \circ A) = A$, because $\iota_A \circ A$ acts as A on the tangent space to a non-singular point of S. Thus, $D : \mathit{Aff}(S) \to GL(2,\mathbb{R})/\pm I$ is a group homomorphism which maps *Aff*(S) onto the Veech group $V(S)$. Assuming S has finite area, $V(S)$ is contained in the group of matrices of determinant ± 1, $PSL^{\pm}(2,\mathbb{R})$, since area is an isometry invariant. Let $V^+(S) = V(S) \cap PSL(2,\mathbb{R})$ denote the subgroup of matrices of determinant 1. There is a totally geodesic immersion of $\mathcal{H}^2/V^+(S)$ into moduli space, obtained as the image of the Teichmüller disk under the natural map from Teichmüller space to moduli space. We will be especially interested in the case when $V(S)$ is a lattice in $PSL^{\pm}(2,\mathbb{R})$. In this case, we say that S has the *lattice property*, and the image of $\mathcal{H}^2/V^+(S)$ in moduli space is known as a *Teichmüller curve*. We will primarily be interested in showing that certain flat structures have the lattice property.

A *saddle connection* is a geodesic segment joining two singularities (cone points) in a flat structure S. A *cylinder decomposition in the direction* θ is a decomposition of S into maximal closed Euclidean cylinders (isometric to $\mathbb{R}/c\mathbb{Z} \times [0,w]$, where c is the *circumference* and w is the width) such that the holonomy vector around each cylinder is parallel to the direction θ. Cylinder decompositions are one of the most useful and fundamental tools in the subject. For example, the Thurston-Veech construction uses pairs of cylinder decompositions each preserved by affine automorphisms whose derivatives are parabolic to generate pseudo-Anosov automorphisms [Thu88] [Vee89].

1.2. **Cylinder graphs.** Our philosophy will utilize a consequence of work of Veech. The lattice property implies the existence of many cylinder decompositions, for the following reason. Teichmüller curves must contain cusps [Vee89]. A cusp corresponds to a conjugacy class of parabolics in the Veech group. Each parabolic preserves a cylinder decomposition in the eigendirection. See [Vee89] and proposition 14 of §2.3.

In this section and whenever we discuss the cylinder intersection graph, we will assume S is a translation surface. Our surface S will have a horizontal cylinder decomposition and a vertical cylinder decomposition. Let $\mathcal{A} = \{\alpha_i\}_{i\in\Lambda}$ and $\mathcal{B} = \{\beta_i\}_{i\in\Lambda}$ be the sets of maximal horizontal and vertical cylinders respectively. (In the general case, $\mathcal{A}$ and $\mathcal{B}$ may have different cardinality and so will need to be indexed by different sets.)

We will associate a *cylinder intersection graph* $\mathcal{G}$ to our cylinder decompositions. Our nodes are the maximal cylinders in the horizontal and vertical directions, $\mathcal{A} \cup \mathcal{B}$. Join an edge between α_i and β_j for every intersection between the two cylinders. Therefore, each edge represents a rectangle with horizontal and vertical sides in our surface. Let $\mathcal{E}$ denote the collection of edges (or rectangles). Define the maps $\alpha : \mathcal{E} \to \mathcal{A}$ and $\beta : \mathcal{E} \to \mathcal{B}$ to be the maps which send an edge $\overline{\alpha_i\beta_j}$ to the nodes α_i and β_j respectively.

Remark 1 (Equivalence to a 2-colored graph)**.** *A* 2-colored graph*, is a graph equipped with a* coloring function C *from the set of nodes,* $\mathcal{V}$*, to* $\{0,1\}$*, with the property that for any two adjancent nodes,* $x, y \in \mathcal{V}$*, we have* $C(x) \neq C(y)$*. The information provided above is equivalent to a 2-colored graph. Simply define* $C(x) = 0$ *if* $x \in \alpha(\mathcal{E}) = \mathcal{A}$ *and* $C(x) = 1$ *if* $x \in \beta(\mathcal{E}) = \mathcal{B}$*. Conversely, the maps* $\alpha, \beta : \mathcal{E} \to \mathcal{V}$ *as well as the decomposition* $\mathcal{V} = \mathcal{A} \sqcup \mathcal{B}$ *are determined by the coloring function.*

We also would like to know how to piece together the rectangles to form our surface. This is determined by knowing when one rectangle lies above another, and when one rectangle lies to the right of another. Let $\mathfrak{e} : \mathcal{E} \to \mathcal{E}$ be the permutation which sends a rectangle to the rectangle which lies to the right (or to the east), and let $\mathfrak{n} : \mathcal{E} \to \mathcal{E}$ be the permutation which sends a rectangle to the rectangle which lies above (or to the north). Note that the rectangle $\mathfrak{e}(e)$ must lie in the same horizontal cylinder as the rectangle e. Thus, we have $\alpha \circ \mathfrak{e} = \alpha$ and $\beta \circ \mathfrak{n} = \beta$. A horizontal cylinder corresponds to an orbit under $\mathfrak{e}$. Thus, for each $a \in \mathcal{A}$, $\mathfrak{e}$ must act transitively on $\alpha^{-1}(a)$. Similarly, $\mathfrak{n}$ must act transitively on $\beta^{-1}(b)$ for each $b \in \mathcal{B}$.

The data consisting of a two-colored graph $\mathcal{G}$ and the edge permutations $\mathfrak{e}$ and $\mathfrak{n}$ determine the combinatorics of our surface as a union of rectangles. To determine the geometry of the surface, it is sufficient to know the width of each cylinder.

1.3. **Lattice examples.** We will now indicate our approach to the Bouw-Möller examples. Consider the (m, n, ∞) triangle group in $PSL(2, \mathbb{R})$, $\Gamma(m, n, \infty)$, as generated by the elements

$$P = \begin{bmatrix} 1 & a+b \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad R = \begin{bmatrix} 0 & -1 \\ 1 & -b \end{bmatrix} \tag{1}$$

where $a = 2\cos\frac{\pi}{m}$ and $b = 2\cos\frac{\pi}{n}$. These matrices satisfy the relations $R^n = (RP)^m = -I$. We will use S to denote a flat structure, and ϕ and ρ will indicate affine automorphisms $S \to S$ with $D\phi = P$ and $D\rho = R$.

P is a parabolic with a horizontal eigenvector, so Veech's proposition implies that in the closed case our surface has a horizontal cylinder decomposition, $\mathcal{A}_{m,n}$. The horizontal direction is sent to the vertical direction by R, so the vertical direction must admit a cylinder decomposition $\mathcal{B}_{m,n}$ as well. The affine automorphism ρ sends maximal horizontal cylinders to maximal vertical cylinders. We will assume that the induced action on maximal cylinders preserves indices. That is, $\rho : \alpha_\lambda \mapsto \beta_\lambda$ for all $\lambda \in \Lambda$. Moreover, R was chosen so that the widths and circumferences of these cylinders are preserved by ρ. We will let w_λ denote the width of the cylinders α_λ and β_λ.

Let $\Lambda = \{(i, j) \in \mathbb{Z}^2 \mid 1 \leq i \leq m-1 \text{ and } 1 \leq j \leq n-1\}$. We use Λ for the indexing sets of $\mathcal{A}_{m,n}$ and $\mathcal{B}_{m,n}$. Let $\mathcal{G}_{m,n}$ be the graph with nodes $\mathcal{A}_{m,n} \cup \mathcal{B}_{m,n}$ formed by joining an edges according to the usual notion adjacency in $\mathbb{Z}^2$. That is, we join an edge between $\alpha_{i,j}$ and $\beta_{i',j'}$ if and only if $(i-i')^2 + (j-j')^2 = 1$ for all $(i, j), (i', j') \in \Lambda$. Note that this makes $\mathcal{G}_{m,n}$ *disconnected*. $\mathcal{G}_{m,n}$ has two components, each of which is the $(m-1, n-1)$ *grid graph*. See figure 1.

The *counter-clockwise ordering* of indices adjacent to (i, j) is the cyclic ordering

$$(i+1, j) \to (i, j+1) \to (i-1, j) \to (i, j-1) \to (i+1, j).$$

And the *clockwise ordering* is the reverse. If any indices in the cyclic order are not in Λ, they are skipped by the ordering. We will follow the following convention to determine the permutations $\mathfrak{e}$ and $\mathfrak{n}$. We number this convention for later reference.

Convention 2. *The map $\mathfrak{e} : \mathcal{E} \to \mathcal{E}$ is determined from a cyclic ordering of the edges with $\alpha_{i,j}$ as an endpoint. As in figure 1, we order edges with endpoint $\alpha_{i,j}$ counter-clockwise when j is odd and clockwise when j is even. Similarly $\mathfrak{n} : \mathcal{E} \to \mathcal{E}$ is determined by a cyclic ordering of edges with $\beta_{i,j}$ as an endpoint. We follow the opposite rule with $\beta_{i,j}$. We order the edges with endpoint $\beta_{i,j}$ clockwise when j is odd and counter-clockwise when j is even.*

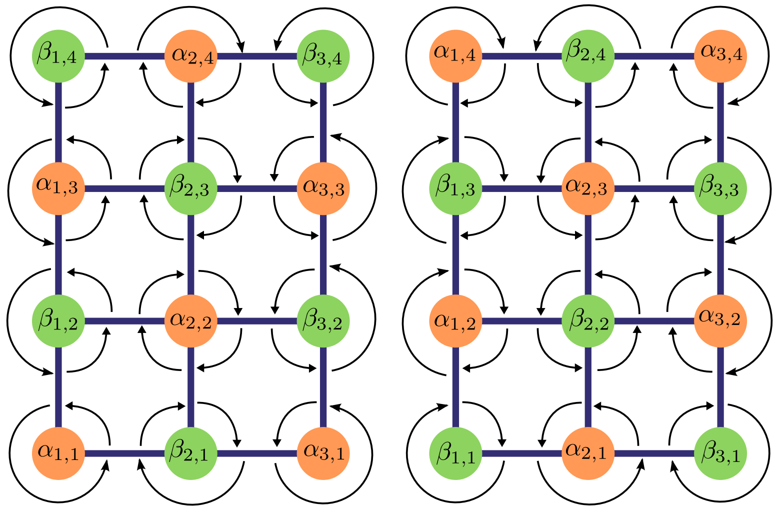

FIGURE 1. The graph $\mathcal{G}_{4,5}$. The edge permutation $\mathfrak{e}$ is indicated by the arrows surrounding the α vertices, and $\mathfrak{n}$ is indicated by the arrows surrounding the β vertices.

Theorem 3 (Lattice surfaces as a union of rectangles)**.** *Let $S_{m,n}$ be the two-component translation surface determined by the graph $\mathcal{G}_{m,n}$, the maps $\mathfrak{e}, \mathfrak{n} : \mathcal{E} \to \mathcal{E}$ as described in convention 2, and the cylinder widths*

$$w_{i,j} = \sin(\frac{i\pi}{m}) \sin(\frac{j\pi}{n}).$$

$S_{m,n}$ has a parabolic automorphism ϕ, with $D\phi = P$, which acts as a single Dehn twist in each horizontal cylinder $\alpha_{i,j}$. $S_{m,n}$ has an affine automorphism ρ, with $D\rho = R$, which sends each cylinder $\alpha_{i,j}$ to $\beta_{i,j}$.

Note that the assignment of the number $w_{i,j}$ to each node $\alpha_{i,j}$ and $\beta_{i,j}$ is a principal eigenvector of either component of the graph $\mathcal{G}_{m,n}$. (A *principal eigenvector* of a graph is an eigenvector corresponding to the eigenvalue of largest magnitude.) We explore why this is true more deeply in section 3.1. See [LM95] for background on eigenvalues and eigenvectors of graphs.

Since ϕ preserves the two components of $S_{m,n}$ and ρ swaps them, we have the following.

Corollary 4. *Each component of $S_{m,n}$ has the lattice property. The Veech group of each component contains the index two subgroup of the (m, n, ∞) triangle group given by $\langle P, R^2, RPR\rangle$.*

Remark 5 (Comparison to Bouw-Möller)**.** *We will show in section 5 that these surfaces $S_{m,n}$ for m and n not both even are precisely the surfaces found by Bouw and Möller* [BM06]*. However when m and n are both even, we will see in corollary 18 that these surfaces do not appear in* [BM06]*.*

The author views the above theorem and corollary as the main point of this discussion. For completeness, we will also compute the complete Veech groups of the individual components. Let $S^1_{m,n}$ denote the component of $S_{m,n}$ which contains the cylinder $\alpha_{1,1}$, and let $S^2_{m,n}$ denote the other component. We will use $S^*_{m,n}$ to denote a component of $S_{m,n}$ when the particular choice is irrelevant. The following two propositions are proved in section 2.2.

Proposition 6. *If m or n is odd, there is a direction preserving isometry $\zeta : S^1_{m,n} \to S^2_{m,n}$.*

Note that this proposition implies that when m or n is odd, we have R in the Veech groups of $S^1_{m,n}$ and $S^2_{m,n}$. This is because we can compose ρ with ζ or ζ^{-1}.

It also turns out that the surfaces $S_{m,n}$ and $S_{n,m}$ are essentially the same.

Proposition 7 (Swapping m and n)**.** *Let Q be the rotation by $\frac{\pi}{2}$, $Q = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. There is an affine map $\eta_{m,n} : S^1_{m,n} \to S^2_{n,m}$ with $D\eta = Q$.*

It also follows that the map $\eta^{-1}_{n,m} : S^2_{m,n} \to S^1_{n,m}$ with derivative $-Q$. Combining the two propositions, we see that when m is odd Q is in the Veech group of $S^*_{m,m}$. When m is even, we see $RQ \in V(S^1_{m,m})$ while $RQ^{-1} \in V(S^2_{m,m})$, since the affine automorphism $\rho : S_{m,m} \to S_{m,m}$ with $D\rho = R$ interchanges the two components.

These arguments turn out to describe the full orientation preserving Veech group of each components $S_{m,n}$. We provide a careful description of these Veech groups in section 2.5.

Remark 8 (Orientation reversing elements)**.** *It follows from proposition 11 (below) that the orientation reversing matrix $\begin{bmatrix} -1 & b \\ 0 & 1 \end{bmatrix}$, for $b = 2\cos\frac{\pi}{n}$, lies in the Veech group of each component of $S^*_{m,n}$.*

1.4. **Decomposition into semiregular polygons.** The (a,b)-*semiregular* $2n$-*gon* is the $2n$-gon whose edge vectors (oriented counterclockwise) are given by

$$\mathbf{v}_k = \begin{cases} a(\cos\frac{i\pi}{n}, \sin\frac{i\pi}{n}) & \text{if } i \text{ is even} \\ b(\cos\frac{i\pi}{n}, \sin\frac{i\pi}{n}) & \text{if } i \text{ is odd} \end{cases}$$

for $i = 0, \ldots, 2n-1$. Denote this $2n$-gon by $P_n(a,b)$. We restrict to the cases where $a, b \geq 0$, but $a \neq 0$ or $b \neq 0$. In the case where one of a or b is zero, $P_n(a,b)$ degenerates to a regular n-gon. We call the edges of $P_n(a,b)$ of length a the *even edges* and the edges of length b the *odd edges*.

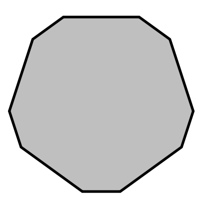

FIGURE 2. The semiregular polygon $P_5(1,2)$.

Fix m and n. Define the polygons $P(k)$ for $k = 0, \ldots, m-1$ by

$$P(k) = \begin{cases} P_n(\sin\frac{(k+1)\pi}{m}, \sin\frac{k\pi}{m}) & \text{if } n \text{ is odd} \\ P_n(\sin\frac{k\pi}{m}, \sin\frac{(k+1)\pi}{m}) & \text{if } n \text{ is even and } k \text{ is even} \\ P_n(\sin\frac{(k+1)\pi}{m}, \sin\frac{k\pi}{m}) & \text{if } n \text{ is even and } k \text{ is odd.} \end{cases} \tag{2}$$

We form a surface by identifying the edges of the polygons in pairs. For k odd, we identify the even sides of $P(k)$ with the opposite side of $P(k+1)$, and identify the odd sides of $P(k)$

with the opposite side of $P(k-1)$. See examples in figures 3 and 4. Similarly, define the polygons $P'(k)$ for $k = 0, \ldots, m-1$ by

$$P'(k) = \begin{cases} P_n(\sin\frac{k\pi}{m}, \sin\frac{(k+1)\pi}{m}) & \text{if } n \text{ is odd} \\ P_n(\sin\frac{(k+1)\pi}{m}, \sin\frac{k\pi}{m}) & \text{if } n \text{ is even and } k \text{ is even} \\ P_n(\sin\frac{k\pi}{m}, \sin\frac{(k+1)\pi}{m}) & \text{if } n \text{ is even and } k \text{ is odd.} \end{cases}$$

For i odd, we identify the even sides of $P'(k)$ with the opposite side of $P'(i-1)$, and identify the odd sides of $P'(k)$ with the opposite side of $P'(k+1)$. Call the disjoint union of these two surfaces $S'_{m,n}$.

Theorem 9 (Semiregular polygon decomposition). *Let* $M = \begin{bmatrix} \csc\frac{\pi}{n} & -\cot\frac{\pi}{n} \\ 0 & 1 \end{bmatrix} \in GL(2,\mathbb{R})$. *There is a direction preserving isometry* $M(S_{m,n}) \to S'_{m,n}$.

The proof of this statement resides in section 4.

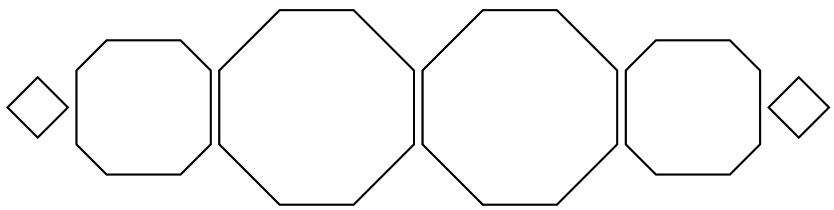

FIGURE 3. These polygons make up one component of the surface $S'_{6,4}$. These are the polygons $P(0)$, $P(1)$, $P(2)$, $P(3)$, $P(4)$ and $P(5)$ from left to right.

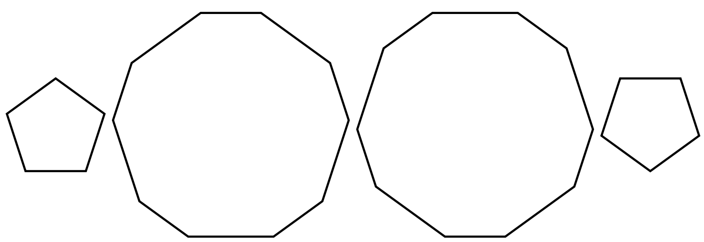

FIGURE 4. These polygons make up one component of the surface $S'_{4,5}$. These are the polygons $P(0)$, $P(1)$, $P(2)$ and $P(3)$ from left to right.

The following proposition will be used to in the proof that we have found all the elements of the Veech group $V^+(S^*_{m,n})$. We use $S'^*_{m,n}$ to denote a component of $S'_{m,n}$ when the particular component doesn't matter.

Proposition 10 (Orthogonal part of the Veech group). *When m or n is odd, $V^+(S'^*_{m,n}) \cap PSO(2,\mathbb{R})$ is a cyclic group of order n. When both m and n are even, $V^+(S'^*_{m,n}) \cap PSO(2,\mathbb{R})$ is a cyclic group of order $n/2$.*

Proof. For notational ease, consider the case $* = 1$. The same argument will hold for $* = 2$. An element of $V^+(S'^1_{m,n}) \cap PSO(2,\mathbb{R})$ must correspond to an affine automorphism which permutes the shortest saddle connections of $S'^1_{m,n}$. The shortest saddle connections of $S'^1_{m,n}$ are the edges of the regular n-gons $P(0)$ and $P(m-1)$. It follows that any orientation preserving affine automorphism from $S'^1_{m,n}$ must send $P(0)$ to either $P(0)$ or $P(m-1)$. Moreover, it is not difficult to see that any isometry from $P(0)$ to $P(0)$ or $P(m-1)$ extends to an affine automorpism of $S'^1_{m,n}$. Note that if m or n is odd, then $P(0)$ differs from $P(m-1)$ by a rotation by π/n. In this case a rotation by π/n generates $V^+(S'^1_{m,n}) \cap PSO(2,\mathbb{R})$. However, when n and m are even, $P(m-1)$ is a translate of $P(0)$. Thus a rotation by $2\pi/n$ generates $V^+(S'^1_{m,n}) \cap PSO(2,\mathbb{R})$. □

It is also worth noting that there are orientation reversing elements of the Veech group.

Proposition 11 (Orientation reversing)**.** *The matrix* $T = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ *is an element of the Veech group of* $S'_{m,n}$*, which preserves each component. Thus* $M^{-1} \circ T \circ M = \begin{bmatrix} -1 & b \\ 0 & 1 \end{bmatrix} \in V(S^*_{m,n})$*, for* $b = 2\cos\frac{\pi}{n}$.

Proof. Each semi-regular polygon as a subset of $\mathbb{R}^2$ is preserved by T up to translation. Moreover, this transformation respects the gluing of $S'_{m,n}$. Thus, this transformation induces an affine automorphism. It follows from theorem 9 that $M^{-1} \circ T \circ M \in V(S^*_{m,n})$. □

2. Combinatorics and affine automorphisms

Recall from section 1.2, that a translation surface with horizontal and vertical cylinder decompositions is determined by the following data.

- A two colored graph $\mathcal{G}$ with vertex set $\mathcal{V} = \mathcal{A} \sqcup \mathcal{B}$ and edge set $\mathcal{E}$. The two coloring is equivalent to the existence of surjective functions $\alpha : \mathcal{E} \to \mathcal{A}$ and $\beta : \mathcal{E} \to \mathcal{B}$, which send each edge to an endpoint of that edge.
- Permutations $\mathfrak{e}, \mathfrak{n} : \mathcal{E} \to \mathcal{E}$ satisfying $\alpha \circ \mathfrak{e} = \alpha$ and $\beta \circ \mathfrak{n} = \beta$. Also, $\mathfrak{e}$ must act transitively on $\alpha^{-1}(a)$ for $a \in \mathcal{A}$, and $\mathfrak{n}$ must act transitively on $\beta^{-1}(b)$ for $b \in \mathcal{B}$.
- A width function $w : \mathcal{V} \to \mathbb{R}_{>0}$.

Moreover, the surface is uniquely determined by this data provided the corresonding horizontal and vertical cylinder decompositions are maximal. Let $S[\mathcal{G}, (\alpha, \beta), (\mathfrak{e}, \mathfrak{n}), w]$. denote the translation surface determined by the provided data.

In the next subsection, we indicate how to recover geometry from this information. In subsection 2.2, we note that the dihedral group of order 8 act nicely on this data.

2.1. Recovering geometry from the graph.

We mentioned in the previous section that the cylinder intersection graph $\mathcal{G}$, which is a 2-colored graph, together with the permutations $\mathfrak{n}, \mathfrak{e} : \mathcal{E} \to \mathcal{E}$ determine the combinatorics of our surface as decomposed into rectangles. We will make this more concrete.

Let $S = S[\mathcal{G}, (\alpha, \beta), (\mathfrak{e}, \mathfrak{n}), w]$ as above. In the general case, the sets of nodes $\mathcal{A} = \alpha(\mathcal{E})$ and $\mathcal{B} = \beta(\mathcal{E})$ may have different indexing sets. Let $\mathcal{A} = \{\alpha_i : i \in \Lambda_a\}$ and $\mathcal{B} = \{\beta_i : i \in \Lambda_b\}$. For each $e = \overline{\alpha_i \beta_j} \in \mathcal{E}$ let R_e be the rectangle $[0, w(\beta_j)] \times [0, w(\alpha_i)] \subset \mathbb{R}^2$. We recover S from the gluing of disjoint rectangles

$$S = \bigsqcup_{e \in \mathcal{E}} R_e / \sim,$$

where $\sim$ identifies the right side of R_e to the left side of $R_{\mathfrak{e}(e)}$ for all $e \in \mathcal{E}$ and identifies the top side of R_e to the bottom side of $R_{\mathfrak{n}(e)}$ for all $e \in \mathcal{E}$.

We include the following as an indication of how to detect the cone singularities from the cylinder intersection graph.

Proposition 12. *Let $e \in \mathcal{E}$. Let $p(e)$ denote the period of e under the commutator map $[\mathfrak{e}, \mathfrak{n}] = \mathfrak{e} \circ \mathfrak{n} \circ \mathfrak{e}^{-1} \circ \mathfrak{n}^{-1} : E \to E$. The cone angle at the lower left vertex of the rectangle R_e is $2\pi p(e)$. Moreover, if $e_0, e_1, \ldots \in \mathcal{E}$ is the orbit of e under $[\alpha, \beta]$ then all lower left vertices of the rectangles $R_{e_0}, R_{e_1}, \ldots$ represent the same point on our surface.*

We say a point is *regular* if the cone angle at the point is 2π. The proposition implies that the lower left vertex of R_e is regular if and only if e is fixed by $[\mathfrak{e}, \mathfrak{n}]$.

Proof. Define $V_{--}(e)$, $V_{+-}(e)$, $V_{-+}(e)$, and $V_{++}(e)$ to be the lower left, lower right, upper left, and upper right vertices of the rectangle corresponding to an edge $e \in \mathcal{E}$. The gluing $\sim$ above implies that we have the following identifications of vertices.

$$\begin{array}{ll} V_{+-}(e) = V_{--}\big(\mathfrak{e}(e)\big) & \qquad V_{++}(e) = V_{-+}\big(\mathfrak{e}(e)\big) \\ V_{++}(e) = V_{+-}\big(\mathfrak{n}(e)\big) & \qquad V_{-+}(e) = V_{--}\big(\mathfrak{n}(e)\big) \end{array}$$

In fact, these are all the identifications of vertices. Hence, the sequence of vertices of rectangles $V_{--}(e)$, $V_{-+}(\mathfrak{n}^{-1}(e))$, $V_{++}(\mathfrak{e}^{-1} \circ \mathfrak{n}^{-1}(e))$, $V_{+-}(\mathfrak{n} \circ \mathfrak{e}^{-1} \circ \mathfrak{n}^{-1}(e))$, $V_{--}(\mathfrak{e} \circ \mathfrak{n} \circ \mathfrak{e}^{-1} \circ \mathfrak{n}^{-1}(e))$ are all identified by $\sim$ and this corresponds to moving clockwise around the image of the point $V_{--}(e)$ on our surface. □

2.2. **Cylinder intersection graphs and the dihedral group D_8.** The dihedral group of order 8, D_8, acts on the plane in a way that preserves the set of directions {horizontal, vertical}. In particular, if S is a translation surface with horizontal and vertical cylinder decompositions and if $A \in D_8$, then the map $A : S \to A(S)$ preserves the collection of all maximal horizontal and vertical cylinders. Thus, there is an action of D_8 on the data associated to the cylinder intersection graph. We record the action below. The proof is simple combinatorics and is left to the reader.

Proposition 13 (Action of D_8)**.** *Let $Q = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ be rotation by $\frac{\pi}{2}$ and $F = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ be a reflection (flip) in the x-axis.*

- $Q\big(S[\mathcal{G}, (\alpha, \beta), (\mathfrak{e}, \mathfrak{n}), w]\big) = S[\mathcal{G}, (\beta, \alpha), (\mathfrak{n}^{-1}, \mathfrak{e}), w]$*, and*
- $F\big(S[\mathcal{G}, (\alpha, \beta), (\mathfrak{e}, \mathfrak{n}), w]\big) = S[\mathcal{G}, (\alpha, \beta), (\mathfrak{e}, \mathfrak{n}^{-1}), w]$.
- $-I\big(S[\mathcal{G}, (\alpha, \beta), (\mathfrak{e}, \mathfrak{n}), w]\big) = S[\mathcal{G}, (\alpha, \beta), (\mathfrak{e}^{-1}, \mathfrak{n}^{-1}), w]$.

For instance, checking that there is an affine map between translation surfaces $\eta : S \to S'$ with $D\eta = Q$ and $S = [\mathcal{G}, (\alpha, \beta), (\mathfrak{e}, \mathfrak{n}), w]$ and $S' = [\mathcal{G}', (\alpha', \beta'), (\mathfrak{e}', \mathfrak{n}), w']$ reduces to a statement about a graph isomorphism. (Assuming our horizontal and vertical cylinder decompositions are maximal.) There is such an η if and only if there is a graph isomorphism $q : \mathcal{G} \to \mathcal{G}'$ such that

(1) $\beta' \circ q = q \circ \alpha$ and $\alpha' \circ q = q \circ \beta$.
(2) $\mathfrak{n}' \circ q = q \circ \mathfrak{e}$ and $\mathfrak{e}' \circ q = q \circ \mathfrak{n}^{-1}$.
(3) $w' \circ q = w$.

Indeed, such a q describes the action of an affine map with derivative Q on the sets of horizontal and vertical cylinders and the rectangles of intersection.

We now apply this to a few problems in our situation. We prove propositions 7 and 6. We use $\mathcal{G}^1_{m,n}$ (resp. $\mathcal{G}^2_{m,n}$) to denote the connected component of $\mathcal{G}_{m,n}$ containing the node $\alpha_{1,1}$ (resp. $\beta_{1,1}$).

Proof of proposition 7. We will show that there is an affine map $\eta : S^1_{m,n} \to S^2_{n,m}$ with $D\eta = Q$. As in the paragraph above, such an η corresponds to a graph isomorphism $q : \mathcal{G}^1_{m,n} \to \mathcal{G}^2_{n,m}$. We consider the map q determined by the following action on nodes as follows.

$$q(\alpha_{i,j}) = \beta_{j,i} \text{ for } i+j \equiv 0 \pmod 2 \quad \text{and} \quad q(\beta_{i,j}) = \alpha_{j,i} \text{ for } i+j \equiv 1 \pmod 2$$

It is a simple check that q satisfies the statements 1-3 above this proof. Thus q induces an affine map $\eta : S^1_{m,n} \to S^2_{n,m}$ with $D\eta = Q$. □

Proof of proposition 6. We prove the proposition in the case of m odd. The case n even follows similarly. We will show that there is an affine map $\eta : S^1_{m,n} \to S^2_{m,n}$ with $D\eta = -I$. This is a direction preserving isometry. Consider the map $q : \mathcal{G}^1_{m,n} \to \mathcal{G}^2_{m,n}$ defined by

$$q(\alpha_{i,j}) = \alpha_{m-i,j} \text{ for } i+j \equiv 0 \pmod 2 \quad \text{and} \quad q(\beta_{i,j}) = \beta_{m-j,i} \text{ for } i+j \equiv 1 \pmod 2$$

We have that

(1) $\alpha \circ q = q \circ \alpha$ and $\beta \circ q = q \circ \beta$.
(2) $\mathfrak{e}^{-1} \circ q = q \circ \mathfrak{e}$ and $\mathfrak{n}^{-1} \circ q = q \circ \mathfrak{n}$.
(3) $w \circ q = w$.

Thus by proposition 13, q extends to a affine map $\eta : S^1_{m,n} \to S^2_{m,n}$ with $D\eta = -I$. □

2.3. **Detecting affine automorphisms via cylinders.** A foundational argument of Veech gave a necessary and sufficient criterion for a parabolic to be in the Veech group. Recall, the *modulus of a cylinder* is the ratio $\frac{width}{circumference}$.

Proposition 14 (Veech [Vee89])**.** *Let S be a closed translation surface. Then there is a parabolic with eigendirection θ in the Veech group $V(S)$ if and only if the following conditions are satisfied.*

- *There is a cylinder decomposition of S in the direction θ.*
- *There is a $d \neq 0$ such that $dm \in \mathbb{Z}$ for each modulus m of a cylinder in the decomposition.*

A few remarks are in order about this proposition from our point of view.

(1) Veech's argument is effective. If the cylinders are horizontal, then $\begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix}$ is in the Veech group.
(2) A powerful feature of this criterion is that it is local to the cylinders, in the sense that it only involves only geometric data about the cylinders.
(3) With the closed condition dropped (ie. in the infinite case), the "if" portion of the proposition still holds.

Now we will provide a criterion for the existence of a affine automorphism of a translation surface S which sends a horizontal maximal cylinder decomposition $\mathcal{A}$ to a vertical maximal cylinder decomposition $\mathcal{B}$.

Lemma 15. *Let S be a translation surface admitting distinct maximal horizontal and vertical cylinder decompositions $\mathcal{A}$ and $\mathcal{B}$ as above. Let Σ_a and Σ_b be the set of saddle connections in the horizontal and vertical directions respectively. Assume $A \in GL(2, \mathbb{R})$ sends the horizontal direction to the vertical direction. Then A is in the Veech group of S if and only if the following holds.*

(1) *The sets* $\mathcal{A}$ *and* $\mathcal{B}$ *have the same cardinality, as do the sets* Σ_a *and* Σ_b.
(2) *There are indexings* $\mathcal{A} = \{\alpha_i\}_{i\in\Lambda}$, $\mathcal{B} = \{\beta_i\}_{i\in\Lambda}$, $\Sigma_a = \{\sigma_j^a\}_{j\in\Lambda'}$ *and* $\Sigma_b = \{\sigma_j^b\}_{j\in\Lambda'}$ *such that the following holds.*
  (a) *There is a direction preserving isometry* $\phi_i : A(\alpha_i) \to \beta_i$.
  (b) *For each* $\sigma_j^a \subset \partial\alpha_i$ *we also have* $\sigma_j^b \subset \partial\beta_i$, *and* $\phi_i : A(\sigma_j^a) \mapsto \sigma_j^b$.

This criterion is "local" in the sense that it only depends on the geometric information about the cylinders and gluing data between cylinders along saddle connections.

Proof. If A is in the Veech group, then there is an affine automorphism ϕ with $D\phi = A$. Choose arbitrary indexings $\mathcal{A} = \{\alpha_i\}_{i\in\Lambda}$ and $\Sigma_a = \{\sigma_j^a\}_{j\in\Lambda'}$. Then use the indexings $\beta_i = \phi(\alpha_i)$ and $\sigma_j^b = \phi(\sigma_j^a)$. The restriction maps $\phi_i = \phi|_{\alpha_i}$ satisfy conditions (a) and (b).

Conversely, suppose conditions (1) and (2) are satisfied. We may rebuild S (up to direction preserving isometry) from the cylinders $\mathcal{A}$ by gluing cylinders along saddle connections in Σ_a as labeled. Similarly, we can recover S from the cylinders $\mathcal{B}$ and the saddle connections Σ_b. The conditions (a) and (b) imply that the isometries ϕ_i respect the gluing relations. Thus, there is a direction preserving isometry $\phi : S \to S$ such that $\phi_i = \phi|_{\alpha_i}$. □

2.4. **Local conditions in the square grid.** The point of this section is to prove theorem 3 from a local point of view via proposition 14 and lemma 15.

We assume that our surface S has horizontal and vertical maximal cylinder decompositions $\mathcal{A}$ and $\mathcal{B}$. We will concentrate on the local picture surrounding the horizontal cylinder α_0 in the left side of figure 5 and the local picture around β_0 on the right side. This figure shows a portion of the cylinder intersection graph $\mathcal{G}$ and also some of the images of edges under the edge permutations $\mathfrak{e}$ and $\mathfrak{n}$. As in §1.3 we assume that that for each i, α_i and β_i have the same circumference and width. Let w_i denote the width of the cylinders α_i and β_i. Let $R = \begin{bmatrix} 0 & -1 \\ 1 & -b \end{bmatrix}$ for some $b \geq 0$. (This is the same R that appears in equation 1 of §1.3).

We will check that under certain conditions, there is a direction preserving isometry $\iota_R : R(\alpha_0) \to \beta_0$. Moreover, ι_R will preserve a natural labeling of saddle connections as required by lemma 15.

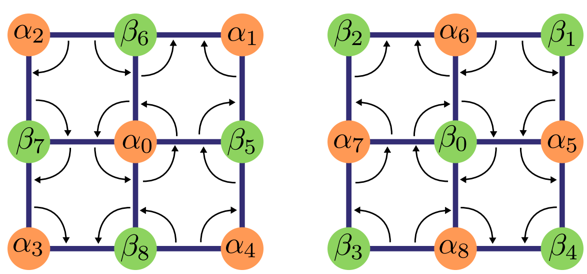

FIGURE 5. The local structure of the graph $\mathcal{G}$ near α_0. Arrows surrounding the α_* vertices determine a portion of the action of $\mathfrak{e} : \mathcal{E} \to \mathcal{E}$. Likewise arrows surrounding the β_* determine a portion of $\mathfrak{n} : \mathcal{E} \to \mathcal{E}$. The rest of the structure of the graph and these maps may be arbitrary.

Lemma 16. *As above, assume that we have the local picture of figure 5. We assume all our widths are non-negative, but allow any of the widths other than* w_0 *to be zero. Let*

$R = \begin{bmatrix} 0 & -1 \\ 1 & -b \end{bmatrix}$ *with* $b \in \mathbb{R}$ *non-negative. If in addition* $w_6 + w_8 = bw_0$*, then there is a direction preserving isometry* $\iota_R : R(\alpha_0) \to \beta_0$ *which sends saddle connections in* $\partial\alpha_0$ *to saddle connections in* $\partial\beta_0$ *according to the rule* $\iota_R : \partial\alpha_0 \cap \partial\alpha_i \mapsto \partial\beta_0 \cap \partial\beta_i$ *for* $i \in \{1, 2, 3, 4\}$.

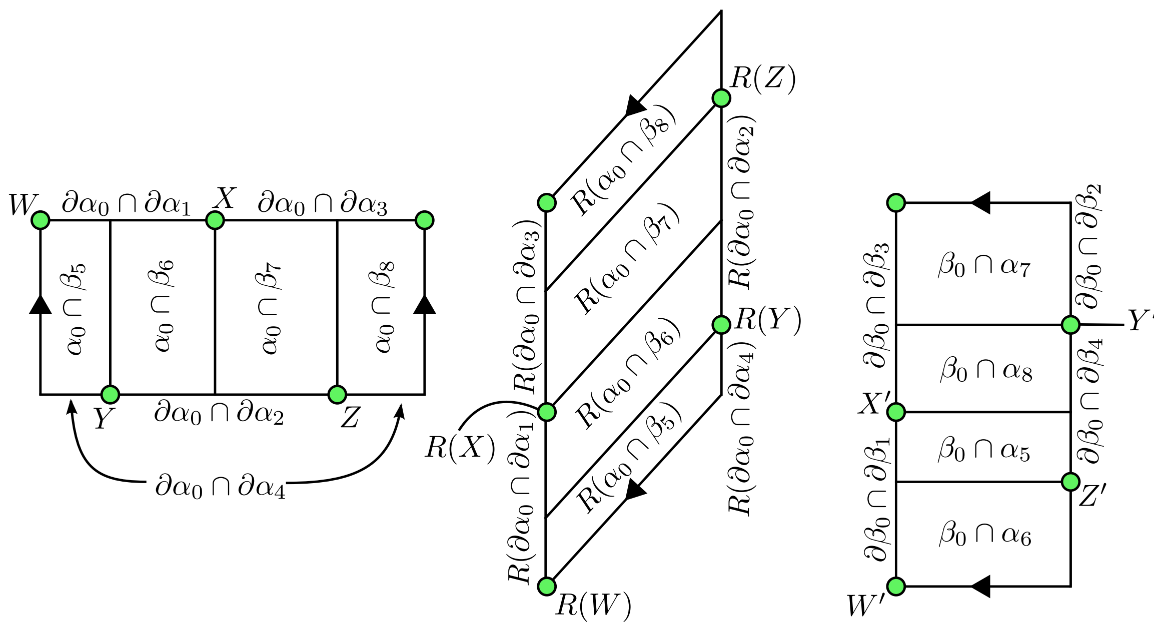

FIGURE 6. From left to right, these are the cylinders α_0, $R(\alpha_0)$, and β_0. The dots denote singularities, which have been given names. The saddle connections are labeled as common boundaries of two cylinders.

Proof. The cylinders α_0 and β_0 have four saddle connections in their boundary as in figure 6. The singular points are indicated in the figure and given names. The unmarked vertices of rectangles are non-singular. This can be determined by using proposition 12 to decide which vertices of rectangles are singularities.

The choice of R guarantees that the width and circumference of α_0 and $R(\alpha_0)$ are the same. Further, β_0 has the same width and circumference as α_0. Since β_0 and $R(\alpha_0)$ are both vertical cylinders, there is a direction preserving isometry between the two. In fact, we may choose the direction preserving isometry $\iota_R : R(\alpha_0) \to \beta_0$ so that it sends $R(W)$ to W'. We will check that this map ι_R also sends $R(X) \mapsto X'$, $R(Y) \mapsto Y'$ and $R(Z) \mapsto Z'$. This will imply that ι_R carries the remaining saddle connections as claimed by the lemma.

The fact that widths of α_i and β_i are the same for $i = 5, \ldots, 8$ guarantees that the saddle connections $\partial\alpha_0 \cap \partial\alpha_i$ and $\partial\beta_0 \cap \partial\beta_i$ have the same length for $i = 1, \ldots 4$. (For instance, $\textit{length}(\partial\alpha_0 \cap \partial\alpha_1) = w_5 + w_6 = \textit{length}(\partial\beta_0 \cap \partial\beta_1)$.) Thus we have $\iota_R(R(X)) = X'$ since $R(\partial\alpha_0 \cap \partial\alpha_3)$ and $\partial\beta_0 \cap \partial\beta_3$ have the same length.

We will now check that $\iota_R(R(Y)) = Y'$. We can see this in terms of vectors. We see that the vector $\overrightarrow{WY} = (w_5, -w_0)$ in α_0. Thus $R(\overrightarrow{WY}) = (w_0, w_5 + bw_0)$ in $R(\alpha_0)$. We also have $\overrightarrow{W'Y'} = (w_0, w_5 + w_6 + w_8)$ in β_0. Thus, we have $R(\overrightarrow{WY}) = \overrightarrow{W'Y'}$ so long as $w_6 + w_8 = bw_0$. Therefore, with the conditions of the lemma true, we have $\iota_R : R(Y) \mapsto Y'$.

We also have that $\iota_R : R(Z) \mapsto Z'$ because the length of the segment $R(\partial\alpha_0 \cap \partial\alpha_2)$ equals the length of $\partial\beta_0 \cap \partial\beta_2$. □

Proof of Theorem 3. Recall $a = 2\cos\frac{\pi}{m}$ and $b = 2\cos\frac{\pi}{n}$. The cylinders are indexed by $\mathbb{Z}^2$ and the widths are given by $w_{i,j} = \sin(\frac{i\pi}{m})\sin(\frac{j\pi}{n})$. Note that these widths are zero just outside the indexing set $\Lambda = \{(i,j) \in \mathbb{Z}^2 : 1 \leq i \leq m-1 \text{ and } 1 \leq j \leq n-1\}$. That is, $w_{i,j} = 0$ if $i = 0$, $i = m$, $j = 0$ or $j = n$.

We first prove that $P = \begin{bmatrix} 1 & a+b \\ 0 & 1 \end{bmatrix}$ is in the Veech group. By proposition 14, it is sufficient to show that the inverse modulus of each cylinder $\alpha_{i,j}$ is $m_{i,j}^{-1} = a+b$. We compute

$$(3) \qquad \begin{aligned} m_{i,j}^{-1} &= \frac{w_{i+1,j}+w_{i,j+1}+w_{i-1,j}+w_{i,j-1}}{w_{i,j}} = \frac{w_{i+1,j}+w_{i-1,j}}{w_{i,j}} + \frac{w_{i,j+1}+w_{i,j-1}}{w_{i,j}} \\ &= \frac{\sin\frac{(i+1)\pi}{m}+\sin\frac{(i-1)\pi}{m}}{\sin\frac{i\pi}{m}} + \frac{\sin\frac{(j+1)\pi}{n}+\sin\frac{(j-1)\pi}{n}}{\sin\frac{j\pi}{n}} = 2\cos\frac{\pi}{m} + 2\cos\frac{\pi}{n} = a+b. \end{aligned}$$

Now we prove that $R = \begin{bmatrix} 0 & -1 \\ 1 & -b \end{bmatrix}$ is in the Veech group. The local picture in the cylinder intersection graph $\mathcal{G}$ surrounding each cylinder $\alpha_{i,j}$ and $\beta_{i,j}$ is identical to that of α_0 and β_0 of figure 5 up to possibly a reflection in a vertical line. Consequently, it is sufficient to check the conditions of 16. We must check that $w_{i,j+1} + w_{i,j-1} = bw_{i,j}$. We have

$$\begin{aligned} w_{i,j+1} + w_{i,j-1} &= \sin\tfrac{i\pi}{m}\left(\sin\tfrac{(j+1)\pi}{n} + \sin\tfrac{(j-1)\pi}{n}\right) = \sin\tfrac{i\pi}{m}\left(2\sin\tfrac{j\pi}{n}\cos\tfrac{\pi}{n}\right) \\ &= 2w_{i,j}\cos\tfrac{\pi}{n} = 2bw_{i,j}. \end{aligned}$$

□

2.5. **The Veech groups of a component.**

We will now describe the complete orientation preserving Veech group of each individual component of $S_{m,n}$.

Theorem 17. *Consider integers m and n with $2 \leq m$, $2 \leq n$, and not $m = n = 2$. Let $S^*_{m,n}$ be either component of $S_{m,n}$.*

- *If $m \neq n$, then*
  - *When m and n are not both even, $V^+(S^*_{m,n})$ is the (m, n, ∞) triangle group*
  $$\langle R, P \mid R^n = (RP)^m = -I\rangle.$$
  - *When m and n are both even, $V^+(S^*_{m,n})$ is the index 2 subgroup of the (m, n, ∞) triangle group given by*
  $$\langle P, R^2, RPR \mid (R^2)^{\frac{n}{2}} = \big((RPR)P\big)^{\frac{m}{2}} = -I\rangle.$$
- *If $m = n$, then*
  - *When m is odd, $V^+(S^*_{m,m})$ is the $(2, m, \infty)$-triangle group*
  $$\langle Q, R \ :\ Q^2 = R^m = -I\rangle.$$
  - *When m is even, $V^+(S^*_{m,m})$ is a $(\frac{m}{2}, \infty, \infty)$ triangle group. We have*
  $$V^+(S^1_{m,m}) = \langle RQ, R^2 \mid (R^2)^{\frac{m}{2}} = -I\rangle \quad \textit{and} \quad V^+(S^2_{m,m}) = \langle RQ^{-1}, R^2 \mid (R^2)^{\frac{m}{2}} = -I\rangle.$$

Proof. Heuristically, fix one of the cases. Let $\Gamma \subset PSL(2, \mathbb{R})$ be the group generated by the elements claimed to generate $V^+(S^*_{m,n})$ in our case of the theorem. We need to show $\Gamma = V^+(S^*_{m,n})$. The comments below show that we have already proved that $\Gamma \subset V^+(S^*_{m,n})$ in each case.

- That P, R^2, and RPR lie in $V^+(S^*_{m,n})$ follows from theorem 3 above, which is proved in section 2.4. Note, the affine automorphism ρ, with $D\rho = R$, interchanges the two components of $S_{m,n}$.
- When m or n is odd, $R \in V^+(S^*_{m,n})$. See the comments below proposition 6.
- Suppose $m = n$. When m is odd, $Q \in V^+(S^*_{m,m})$. When m even, we have $RQ \in V^+(S^1_{m,m})$ and $RQ^{-1} \in V^+(S^2_{m,m})$. See the remarks below proposition 7.

We also must show that $V^+(S^*_{m,n}) \subset \Gamma$. Given that $\Gamma \subset V^+(S^*_{m,n})$, there is a covering map $\mathbb{H}^2/\Gamma \to \mathbb{H}^2/V^+(S^*_{m,n})$. These quotients of the hyperbolic plane are all hyberbolic orbifold with the topology of a punctured sphere (with cone singularities).

Let M be an orbifold which is topologically a 2-sphere (possibly with punctures). Let S be the set of singularities of M. That is, S is the collection of cone points and punctures of M. In this specific case, the *Euler number* of M is given by the formula

$$\chi(M) = 2 + \sum_{s \in S} (\frac{1}{|G_s|} - 1),$$

where G_s is the group associated to the singularity s. Treat $1/|G_s| = 0$ if G_s is infinite, ie. when s is a puncture. For more information on the Euler number of an orbifold see chapter 13 of [Thu81]. Note that a hyperbolic orbifold must have negative Euler number. Moreover, if $M \to N$ is a covering map of degree d, then $\chi(N) = \chi(M)/d$. In particular, we have $\chi(M) \leq \chi(N)$ with equality implying that $M = N$. Note further that adding more singular points only lowers the Euler number. We apply this argument to the case $M = \mathbb{H}^2/\Gamma$ and $N = \mathbb{H}^2/V^+(S^*_{m,n})$.

We know that N has at least one puncture, corresponding to the horizontal cylinder decomposition. If n and m are even, N must have another puncture corresponding to the vertical cylinder decomposition. Here, no element of $V^+(S^*_{m,n})$ may send the horizontal direction to the vertical direction. This is because when m and n are even, the number of maximal horizontal and vertical cylinders of $S_{m,n}$ differ by one.

Now we consider the finite order singularities. Call an elliptic $e \in V^+(S^*_{m,n})$ maximal if any elliptic that commutes with it has smaller order. When n or m is odd, the matrices $R, RP \in \Gamma$ are elliptics of order n and m respectively. Moreover, they are maximal in $V^+(S_{m,n})$, by proposition 10. Thus, so long as $m \neq n$, we have two singularities of order n and m. When $m = n$ is odd we have at least one singularity of order $m = n$. If both n and m are even, then by proposition 10 we have maximal elliptics of orders $n/2$ and $m/2$, given by R^2 and $(RP)^2$. When $n \neq m$ this yields at least two singularities, and when $n = m$ this yields at least one singularity.

The argument above tells us that N has at least the same number of singularities as M for all cases of n and m. Moreover, it has singularities of the same orders. Thus we have $\chi(M) \geq \chi(N)$. Since we have a covering $M \to N$, it must be that $\chi(M) = \chi(N)$. Thus $M = N$ and $\Gamma = V^+(S^*_{m,n})$. □

The following corollary indicates that the surfaces $S^1_{m,n}$ for m and n even seem to be new examples of surfaces with the lattice property. Note that the surfaces described in [BM06] have Veech groups which are triangle groups.

Corollary 18. *The surface $S^1_{m,n}$ with both m and n even has a Veech group which is not a triangle group.*

3. Eigenvectors of graphs

3.1. **An algebraic interpretation.** In this section we suppose we are in the following situations. S is a translation surface admitting maximal cylinder decompositions $\mathcal{A}$ and $\mathcal{B}$ in the horizontal and vertical directions respectively. Assume that each is indexed by the same set Λ and that there is an affine automorphism ρ such that $\rho(\alpha_i) = \beta_i$ for all $i \in \Lambda$. Moreover, we assume that the width and circumference of each cylinder α_i and β_i are the same. Denote these widths and circumferences as w_i and c_i respectively.

Let J be the *cylinder intersection matrix*. That is, J is the matrix of entries with entry $J_{i,j}$ equal to the number of intersection between α_i and β_j. Let $\mathbf{w}$ denote the vector of widths and $\mathbf{c}$ denote the vector of circumferences. Then we have that

$$circumference(\beta_i) = \sum_{j\in\Lambda} \#(\beta_i \cap \alpha_j) width(\alpha_j). \tag{4}$$

This corresponds to the matrix equation $\mathbf{c} = J\mathbf{w}$. By similar logic with the roles of α and β reversed, we have $\mathbf{c}^T = \mathbf{w}^T J$. Therefore, we have that

$$J\mathbf{w} = J^T\mathbf{w}. \tag{5}$$

In some sense, J acts symmetrically on $\mathbf{w}$. Now let M denote the diagonal matrix of moduli of cylinders. That is, let $M_{i,i} = \frac{w_i}{c_i}$. Then we have the second matrix equation $M\mathbf{c} = \mathbf{w}$. Therefore, the widths of cylinder can be viewed as a positive eigenvector corresponding to the eigenvalue 1 by the matrix equation

$$MJ\mathbf{w} = \mathbf{w} \tag{6}$$

We wish to relate $\mathbf{w}$ to eigenvectors and eigenvalues on graphs. *Eigenvectors* and *eigenvalues* of a graph are simply the eigenvectors and eigenvalues of the graph's adjacency matrix. See [LM95] for more information. Let $\mathcal{G}'$ denote the quotient $\mathcal{G}/\sim$ which identifies the nodes $\alpha_i \sim \beta_i$ for all $i \in \Lambda$. Call the identified node γ_i. The *adjacency matrix* of $\mathcal{G}'$ is the matrix A whose entry $A_{i,j}$ is the number of edges between γ_i and γ_j. In fact, this implies that $A = J + J^T$. By equations 5 and 6, we see $\mathbf{w}$ satisfies $MA\mathbf{w} = 2\mathbf{w}$. When all the moduli are the same, we see that the common inverse modulus of the cylinders is half the Perron-Frobenius eigenvalue of A and $\mathbf{w}$ is the corresponding eigenvector.

Now restrict to our case of $S_{m,n}$. All our cylinders have the same moduli. The edges of the quotient graph $\mathcal{G}'_{m,n}$ come in pairs joining the same nodes. Let $\mathcal{G}''_{m,n}$ be the quotient of $G'_{m,n}$ which identifies these pairs of edges. The resulting graph $\mathcal{G}''_{m,n}$ is graph-isomorphic to one of the components of $\mathcal{G}_{m,n}$.

The *Cartesian product of two graphs* $\mathcal{G}$ and $\mathcal{H}$ is the graph with nodes given by pairs (g, h) where g is a node of $\mathcal{G}$ and h is a node of $\mathcal{H}$. We join an edge between (g, h_1) and (g, h_2) for all g and all edges $\overline{h_1h_2}$ in $\mathcal{H}$ and join an edge between (g_1, h) and (g_2, h) for all h and all edges $\overline{g_1g_2}$ in $\mathcal{G}$. The adjacency matrix of the Cartesian product of two graphs is closely related to the adjacency matrix of the original graphs. Let $A(\mathcal{G})$ denote the adjacency matrix of the graph G. Then $A(\mathcal{G} \times \mathcal{H}) = A(\mathcal{G}) \times A(\mathcal{H})$, where $\times$ denotes the Cartesian product of two matrices. See [LY93] for more information.

Proposition 19 (Eigenvectors and products of graphs)**.** *Assume the weights w_g for nodes of $\mathcal{G}$ form an eigenvector corresponding to the eigenvalue λ_1 for the adjacency matrix of the graph $\mathcal{G}$, and the weights w_h form an eigenvector corresponding to the eigenvalues λ_2 for the adjacency matrix of the graph $\mathcal{H}$. Then the weights $w_{g,h} = w_g w_h$ for the nodes of $\mathcal{G} \times \mathcal{H}$*

form an eigenvector corresponding to the eigenvalue $\lambda_1 + \lambda_2$ of the adjacency matrix of the graph $\mathcal{G} \times \mathcal{H}$. The weights assigned to nodes of the fibers of the projections $\mathcal{G} \times \mathcal{H} \to \mathcal{G}$ and $\mathcal{G} \times \mathcal{H} \to \mathcal{H}$ are constant multiples of the eigenvectors provided for $\mathcal{H}$ and $\mathcal{G}$ respectively.

The proof of the proposition trivially follows from the fact that $A(\mathcal{G} \times \mathcal{H}) = A(\mathcal{G}) \times A(\mathcal{H})$. See [LY93].

Let $\mathcal{L}_n$ denote the line of nodes $1, 2, \ldots, n-1$, with an edge joining each integer to the subsequent integer in the list. The adjacency matrix of $\mathcal{L}_n$ has the Perron-Frobenius eigenvector given by the weights $w_i = \sin \frac{i\pi}{n}$ corresponding to the eigenvalue $\cos \frac{\pi}{n}$.

We remark that a surprising coincidence has occurred. The graph $\mathcal{G}''_{m,n}$ may be seen to be graph isomorphic to the product $\mathcal{L}_m \times \mathcal{L}_n$. Therefore, the widths of cylinders are forced to be as in the theorem 3 by the algebraic considerations above. On the other hand, the geometry of lemma 16 states local conditions on the widths of cylinders. Algebraically, these conditions are precisely equivalent to locally checking that the assignments of cylinder widths satisfy the eigenvector conditions on the fibers of the projection maps $\mathcal{G}''_{m,n} \to \mathcal{L}_m$ and $\mathcal{G}''_{m,n} \to \mathcal{L}_n$ as described in proposition 19.

3.2. **Infinite translation surfaces.** In this section, we construct new examples of infinite translation surfaces with the lattice property. We begin by generalizing our discussion of the adjacency matrix of a graph.

Let $\mathcal{G}$ be a graph, which we assume to be connected but have no multiple edges or loops. Let $\mathcal{V}$ denote the vertex set, which is possibly countably infinite. Let $\mathcal{E}(x) \subset \mathcal{V}$ denote the vertices adjacent to a vertex $x \in \mathcal{V}$. We assume each $\mathcal{E}(x)$ is finite. Let $\mathbb{C}^{\mathcal{V}}$ denote the set of functions $f : \mathcal{V} \to \mathbb{C}$. We define the adjacency operator $H : \mathbb{C}^{\mathcal{V}} \to \mathbb{C}^{\mathcal{V}}$ by

$$(Hf)(x) = \sum_{y \in \mathcal{E}(x)} f(y).$$

This is a generalization of multiplication by the adjacency matrix to infinite graphs. An eigenfunction of H corresponding to the eigenvalue $\lambda \in \mathbb{C}$ is an $f \in \mathbb{C}^{\mathcal{V}}$ such that $Hf = \lambda f$.

Let $\mathcal{L}_{\mathbb{Z}}$ denote denote the graph with integer vertices whose edges consist of pairs of integers whose difference is ± 1. Let $\mathcal{G}$ be a connected subgraph of $\mathcal{L}_{\mathbb{Z}}$ with vertex set $\mathcal{V}$. Up to translation and reflection, we can assume $\mathcal{V} = \mathbb{Z}$, $\mathcal{V} = \mathbb{N}$, or $\mathcal{V} = \{1, \ldots, n-1\}$.

Definition 20. *With $\mathcal{G}$ and $\mathcal{V}$ as in the previous paragraph, the* critical eigenfunctions *of $H : \mathbb{C}^{\mathcal{V}} \to \mathbb{C}^{\mathcal{V}}$ are the eigenfunctions*

- *$f(x) = 1$ if $\mathcal{V} = \mathbb{Z}$ (corresponding to $\lambda = 2$),*
- *$f(x) = x$ if $\mathcal{V} = \mathbb{N}$ (corresponding to $\lambda = 2$), and*
- *$f(x) = \sin \frac{x\pi}{n}$ if $\mathcal{V} = \{1, \ldots, n-1\}$ (corresponding to $\lambda = \cos \frac{\pi}{n}$).*

We remark, without proof, that these eigenfunctions are the unique positive eigenfunctions up to scalar multiplication determined by these values of λ. In addition, the chosen value of λ is the smallest such that there are corresponding positive eigenfunctions.

We now repeat the construction described in section 1.3 for Cartesian products of infinite graphs of connected subgraphs. Let $\mathcal{I}$ and $\mathcal{J}$ be connected subgraphs of $\mathcal{L}_{\mathbb{Z}}$, with vertex sets $\mathcal{V}_{\mathcal{I}}$ and $\mathcal{V}_{\mathcal{J}}$. We may assume that $\mathcal{I}, \mathcal{J} \in \{\mathcal{L}_n \ : \ n \geq 2\} \cup \{\mathcal{L}_{\mathbb{N}}, \mathcal{L}_{\mathbb{Z}}\}$. Let $\mathcal{G}$ be the graph which is the union of two copies of the graph $\mathcal{I} \times \mathcal{J}$. The vertices $\mathcal{V}$ of $\mathcal{G}$ are the set

$$\mathcal{V} = \mathcal{A} \cup \mathcal{B}, \text{ where } \mathcal{A} = \{\alpha_{i,j} \ : \ i \in \mathcal{I} \text{ and } j \in \mathcal{J}\} \text{ and } \mathcal{B} = \{\beta_{i,j} \ : \ i \in \mathcal{I} \text{ and } j \in \mathcal{J}\}.$$

The graph $\mathcal{G}$ is constructed by joining $\alpha_{i,j}$ to $\beta_{i',j'}$ for each pair with $(i-i')^2+(j-j')^2=1$. Let $\mathcal{E}$ be the edges of this graph, and define the permutations $\mathfrak{e}, \mathfrak{n} : \mathcal{E} \to \mathcal{E}$ as in convention 2.

The above paragraph builds the combinatorial structure for a translation surface built as a union of rectangles, coming from the intersections of cylinders in horizontal and vertical cylinder decompositions. Choosing the widths of these cylinders (which correspond to the vertices of the graph $\mathcal{G}$) determines the flat structure. Let $f_{\mathcal{I}} : \mathcal{V}_{\mathcal{I}} \to \mathbb{R}$ and $f_{\mathcal{J}} : \mathcal{V}_{\mathcal{J}} \to \mathbb{R}$ be the critical eigenfunctions. Let $\lambda_{\mathcal{I}}$ and $\lambda_{\mathcal{J}}$ denote the critical eigenvalues. Let $S_{\mathcal{I},\mathcal{J}}$ be the surface determined by the width function

$$w(\alpha_{i,j}) = w(\beta_{i,j}) = f_{\mathcal{I}}(i) f_{\mathcal{J}}(j).$$

Then $S_{\mathcal{I},\mathcal{J}}$ is an infinite translation surface with two components.

Theorem 21. *The surface $S_{\mathcal{I},\mathcal{J}}$ for have the lattice property. The Veech group is generated by*

$$P = \begin{bmatrix} 1 & \lambda_{\mathcal{I}} + \lambda_{\mathcal{J}} \\ 0 & 1 \end{bmatrix} \quad \textit{and} \quad R = \begin{bmatrix} 0 & -1 \\ 1 & -\lambda_{\mathcal{J}} \end{bmatrix}.$$

The proof is the same as the proof of theorem 3, because the proof only relies on proposition 14 and lemma 16, which are both local conditions.

We describe decompositions of these surfaces into analogs of semiregular polygons in section 6.

4. Gluings of semiregular polygons

In this section we prove theorem 9, which provides a decomposition of the surface $M(S_{m,n})$ into semiregular polygons.

Proof of theorem 9. We will decompose the surface $S_{m,n}$ into polygons. Then we will show that the image of these polygons under M are the desired semiregular polygons. The proof will end by noting that the edges are identified as described above the statement of theorem 9.

We claim it is sufficient to prove existence of a direction preserving isometry from a single component $M(S_{m,n})$ to a single component of $S'_{m,n}$. Let $S \equiv T$ denote the existence of a direction preserving isometry between the translation surfaces S and T. The automorphism ρ of $S_{m,n}$ defined in theorem 3 swaps the two components of $S_{m,n}$. Let S_1 denote one of the components of $S_{m,n}$. Then, $S_{m,n} \equiv S_1 \sqcup R(S_1)$ where $R = D\rho$ as in theorem 3. Let S'_1 denote one of the components of $S'_{m,n}$. Then if $M(S_1) \equiv S'_1$, we have that $MR(S_1) \equiv R'M(S_1) \equiv R'(S'_1)$, where $R' = MRM^{-1}$. But a computation reveals that this R' is just a Euclidean rotation by $(1 + \frac{1}{n})\pi$. Such a rotation is the derivative of an affine automorphism of $S'_{m,n}$ which interchanges the two components. Thus, we have found the desired direction preserving isometry which sends the second component of $S_{m,n}$ to the second component of $S'_{m,n}$.

Let S_1 denote the component of $S_{m,n}$ containing the cylinder $\alpha_{1,1}$. Similarly, let $\mathcal{G}_1$ denote the connected component of $\mathcal{G}_{m,n}$ containing the node $\alpha_{1,1}$. The nodes of $\mathcal{G}_1$ consists of those horizontal cylinders $\alpha_{i,j}$ with $i+j$ even, and those vertical cylinders $\beta_{i,j}$ with $i+j$ odd.

For ease of exposition, we consider the augmented graph $\mathcal{G}'_1$ obtained by attaching *degenerate nodes* and *degenerate edges*. The nodes of G_1 are in bijection with the coordinates $(i,j) \in \mathbb{R}^2$ with $0 < i < m$ and $0 < j < n$. The nodes of $\mathcal{G}'_1$ will be in bijection with those

$(i, j) \in \mathbb{R}^2$ with $0 \leq i \leq m$ and $0 \leq j \leq n$. Our added nodes are called *degenerate nodes*. We follow the same naming conventions for these nodes. We join new *degenerate edges* between nodes of distance one. An example graph is shown in figure 7. We call a degenerate edge e α-degenerate, β-degenerate or completely degenerate if ∂e contains a degenerate α-node, a degenerate β-node or both, respectively. We also define permutations $\mathfrak{e}', \mathfrak{n}' : \mathcal{E}(\mathcal{G}_1') \to \mathcal{E}(\mathcal{G}_1')$ following convention 2.

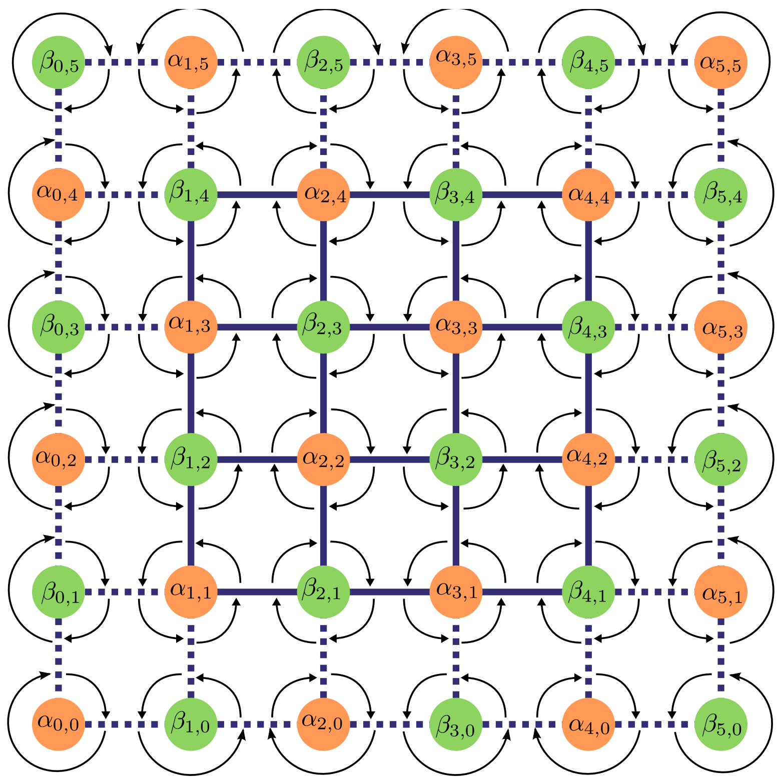

FIGURE 7. The augmented graph $\mathcal{G}_1'$ for the connected component $\mathcal{G}_1 \subset \mathcal{G}_{5,5}$. The degenerate edges are drawn as dotted lines. The map $\mathfrak{e}'$ is given by the arrows surrounding the α vertices, and the map $\mathfrak{n}'$ is given by the arrows surrounding the β vertices.

These *degenerate edges* correspond to degenerate rectangles on our surface S_1. A *degenerate rectangle* is a rectangle with zero width or zero height. (The added nodes correspond to cylinders with zero width according to the formula given in theorem 3.) The α-degenerate edges correspond to horizontal saddle connections (rectangles with zero height) and the β-degenerate edges correspond to vertical saddle connections. The completely degenerate edges correspond to points on our surface.

Each edge e in the graph $\mathcal{G}_1'$ corresponds to a rectangle (or degenerate rectangle) $R_e = R(e)$ in the surface S_1 with horizontal and vertical sides. The *positive diagonal* of a rectangle with horizontal and vertical sides is the diagonal with positive slope. Let $\mathbf{d}(e)$ denote the vector which points along the negative diagonal, oriented rightward and upward. The lower triangle, denoted $L(e)$, of a rectangle $R(e)$ is the triangle below the positive diagonal. The upper triangle, $U(e)$ is the triangle above the positive diagonal. See figure 8. For degenerate rectangles, we take $R(e) = L(e) = U(e)$ to be the corresponding saddle connection, or point.

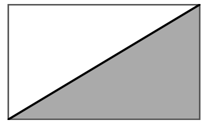

FIGURE 8. A rectangle's positive diagonal. The lower triangle is shaded gray, and the upper triangle is white.

Recall that $\mathcal{G}_1'$ is naturally embedded in $\mathbb{Z}^2$. Let $c_{i,j}$ denote the node of $\mathcal{G}_1'$ in the position (i,j). That is

$$c_{i,j} = \begin{cases} \alpha_{i,j} & \text{if } i+j \text{ is even} \\ \beta_{i,j} & \text{if } i+j \text{ is odd.} \end{cases}$$

We now define our decomposition of S_1 into polygons. Let H_k denote the set of edges of G_1', $H_k = \{\overline{c_{k,i}c_{k+1,i}} \ : \ 0 < i < m\}$ for $k = 0, \dots, n-1$. ($\bigcup_k H_k$ is the set of horizontal edges in the graph $\mathcal{G}_1'$, and the edges in each H_k lie in a column.) For $k = 0, \dots, n-1$ define the polygon $Q(k) \subset S_1$ by

$$Q(k) = \bigcup_{e \in H_k} R(e) \cup L\big(\mathfrak{n}'(e)\big) \cup L\big(\mathfrak{e}'^{-1}(e)\big) \cup U\big(\mathfrak{n}'^{-1}(e)\big) \cup U\big(\mathfrak{e}'(e)\big). \tag{7}$$

An example decomposition is shown in figure 9.

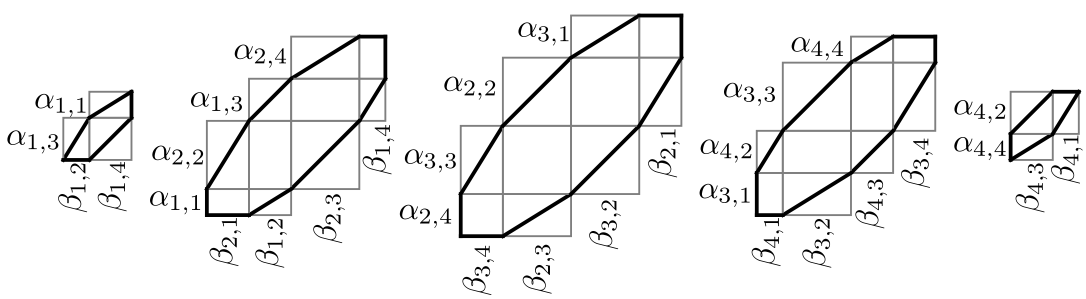

FIGURE 9. The surface $S_{5,5}$ decomposes into the polygons $Q(0), Q(1), \dots Q(4)$ ordered from left to right. Portions of the horizontal cylinders, α_*, and the vertical cylinders, β_* are labeled.

We will show that these subsets $Q(k)$ are in fact polygons. Suppose k is odd and $0 < k \leq n-1$. Define the edge $e_i = \overline{c_{k,i}c_{k+1,i}}$ for $i = 0, \dots, m$. We have $e_i \in H_k$ when $i = 1, \dots, m-1$. We have that $\mathfrak{n}' \circ \mathfrak{e}'(e_i) = \mathfrak{e}' \circ \mathfrak{n}'(e_i) = e_{i+1}$. Therefore many of the triangles are mentioned twice in equation 7 (e.g. $L\big(\mathfrak{n}'(e_1)\big) = L\big(\mathfrak{e}'^{-1}(e_2)\big)$.) Moreover, the top right coordinate vertex $R(e_i)$ is the same as the bottom left vertex of $R(e_{i+1})$ and this point is non-singular provided neither rectangle is degenerate. Thus this point is non-singular for $i = 1, \dots, n-2$. With this in mind, we see that $Q(k)$ is formed by a chain of rectangles $R(e_i)$ moving to the northeast with some triangles added on. We define the edge vectors of $Q(k)$ to be $\mathbf{w}_i$ for

$i = 0, \ldots, 2m-1$ by

$$\mathbf{w}_i = \begin{cases} \mathbf{d}(\overline{\alpha_{k+1,i}\beta_{k+1,i+1}}) & \text{if } i < n \text{ and } i \text{ even} \\ \mathbf{d}(\overline{\alpha_{k,i}\beta_{k,i+1}}) & \text{if } i < n \text{ and } i \text{ odd} \\ -\mathbf{d}(\overline{\beta_{k+1,2n-1-i}\alpha_{k+1,2n-i}}) & \text{if } i \geq n \text{ and } i \text{ even} \\ -\mathbf{d}(\overline{\beta_{k,2n-1-i}\alpha_{k,2n-i}}) & \text{if } i \geq n \text{ and } i \text{ odd} \end{cases} \tag{8}$$

Therefore we have

$$\mathbf{w}_i = \begin{cases} \sin\frac{(k+1)\pi}{m}(\sin\frac{(i+1)\pi}{n}, \sin\frac{i\pi}{n}) & \text{if } i \text{ is even} \\ \sin\frac{k\pi}{m}(\sin\frac{(i+1)\pi}{n}, \sin\frac{i\pi}{n}) & \text{if } i \text{ is odd} \end{cases}$$

for $i = 0, \ldots, 2n-1$. By a simple trigonometric calculation, we have that

$$M\mathbf{w}_i = \begin{cases} \sin\frac{(k+1)\pi}{m}(\cos\frac{i\pi}{n}, \sin\frac{i\pi}{n}) & \text{if } i \text{ is even} \\ \sin\frac{k\pi}{m}(\cos\frac{i\pi}{n}, \sin\frac{i\pi}{n}) & \text{if } i \text{ is odd.} \end{cases}$$

Thus, $M(Q(k)) = P_n(\sin\frac{(k+1)\pi}{m}, \sin\frac{k\pi}{m})$ is the same polygon as $P(k)$.

The case of k even is similar. Let $e_i = \overline{c_{k,i}c_{k+1,i}}$ for $i = 0, \ldots, m$. We have $\mathfrak{n}'^{-1} \circ \mathfrak{e}'^{-1}(e_i) = \mathfrak{e}'^{-1} \circ \mathfrak{n}'^{-1}(e_i) = e_{i+1}$. So, again the lower left and top right vertices are non-singular. But, the chain of rectangles $R(e_i)$ moves toward the southwest. The edges are determined by the added edges. Using $i = n + j$, we have the edge vectors $\mathbf{w}_i = \mathbf{w}_{n+j}$ as follows.

$$\mathbf{w}_{n+j} = \begin{cases} -\mathbf{d}(\overline{\alpha_{k,j}\beta_{k,j+1}}) & \text{if } j \geq 0 \text{ and } j \text{ even} \\ -\mathbf{d}(\overline{\alpha_{k+1,j}\beta_{k+1,j+1}}) & \text{if } j \geq 0 \text{ and } j \text{ odd} \\ \mathbf{d}(\overline{\beta_{k,-j-1}\alpha_{k,-j}}) & \text{if } j < 0 \text{ and } j \text{ even} \\ \mathbf{d}(\overline{\beta_{k+1,-j-1}\alpha_{k+1,-j}}) & \text{if } j < 0 \text{ and } j \text{ odd.} \end{cases} \tag{9}$$

We see

$$\mathbf{w}_{n+j} = \begin{cases} -\sin\frac{k\pi}{m}(\sin\frac{(j+1)\pi}{n}, \sin\frac{j\pi}{n}) & \text{if } j \text{ is even} \\ -\sin\frac{(k+1)\pi}{m}(\sin\frac{(j+1)\pi}{n}, \sin\frac{j\pi}{n}) & \text{if } j \text{ is odd.} \end{cases}$$

So, we have

$$M\mathbf{w}_i = M\mathbf{w}_{n+j} = \begin{cases} \sin\frac{k\pi}{m}(\cos\frac{i\pi}{n}, \sin\frac{i\pi}{n}) & \text{if } i - n = j \text{ is even} \\ \sin\frac{(k+1)\pi}{m}(\cos\frac{i\pi}{n}, \sin\frac{i\pi}{n}) & \text{if } i - n = j \text{ is odd.} \end{cases}$$

Thus, $M(Q(k)) = P_n(\sin\frac{k\pi}{m}, \sin\frac{(k+1)\pi}{m})$ when n is even and $M(Q(k)) = P_n(\sin\frac{(k+1)\pi}{m}, \sin\frac{k\pi}{m})$ when n is odd. In either case, we have $M(Q(k)) = P(k)$.

Finally, we note that the gluings of polygons agree with the gluing definition given in section 1.4. A comparison between equations 8 and 9 reveals that, for k odd, the even sides of $Q(k)$ are identified to the opposite side of $Q(k-1)$ and the odd sides of $Q(k)$ are identified to the opposite side of $Q(k+1)$. □

5. Comparison to the Bouw-Möller surfaces

In this section, we will show that the surfaces discovered by Bouw and Möller are the same as the surfaces we have described so long as m and n are not both even. The following appears in theorem 5.15 of [BM06].

Theorem 22 (Bouw-Möller)**.** *Suppose m and n are odd and relatively prime. The surface X_0 together with the differential ω_0 given below determines a translation surface with an affine automorphism group which is an (m, n, ∞) triangle group.*

$$X_0 : \quad y^{2n} = (u-2) \prod_{j=1}^{(m-1)/2} \left(u - 2\cos\frac{2j\pi}{m}\right)^2$$

$$\omega_0 = \frac{y \; du}{(u-2)\prod_{j=1}^{(m-1)/2}(u - 2\cos\frac{2j\pi}{m})}$$

Remark 23. *The condition that m and n must be relatively prime appears in* [BM06]*, but may be removed (for both theorems 22 and 25, above and below). The only change is the number and orders of the singularities of the translation surface associated to the pair (X_0, ω_0). In all cases, the singularities arise only when $u = \infty$.*

The following says that the surface described above is the same as the surface we built with semiregular polygons.

Proposition 24. *Up to scaling, there is an isometry from the translation surface determined by the pair (X_0, ω_0) in theorem 22 and one component of the surface $S'_{m,n}$ defined in subsection 1.4.*

Proof. First let us understand the surface X_0. Consider the map $u : X_0 \to \hat{\mathbb{C}}$ given by the u-coordinate. This map is $2n : 1$ except at the points $u = 2$ and $u = \infty$, where it is $1 : 1$, and the points $u = 2\cos\frac{2j\pi}{m}$, where it is $2 : 1$. The surface X_0 and the differential ω_0 have the rotational and reflective symmetry of a regular $2n$-gon. We may separate the u-plane into strips and half-planes which each contain one of the points $u = 2$ or $u = 2\cos\frac{2j\pi}{m}$ by cutting along vertical lines. We can pull these strips and half-planes back to the translation surface (X_0, ω_0). The pullback of the half-plane containing $u = 2$ to (X_0, ω_0) has the symmetries of a regular $2n$-gon, but the pullback of the boundary of the strip is not geodesic. Similarly, each component of the pull back of the strip containing $u = 2\cos\frac{2j\pi}{m}$ for $1 \leq j < \frac{m-1}{2}$ has the symmetries of semiregular $2n$-gon. The edges of the pullback alternate between pullbacks of the two edges of the strip. For $j = \frac{m-1}{2}$ the left half plane pulls back to two components with the symmetries of a regular n-gon. Then, we will show that the boundary curves of these regions can be straightened to geodesics. We only alter this argument by replacing vertical lines with carefully chosen hyperbolas.

Let $u = 2\cos z$. Let d denote the denominator of the expression for ω_0. Then by trigonometric manipulations we have

$$d = (u-2) \prod_{j=1}^{(m-1)/2} (u - 2\cos\frac{2j\pi}{m}) = -4\sin\frac{mz}{2}\sin\frac{z}{2}.$$

We also have

$$(u-2) \prod_{j=1}^{(m-1)/2} \left(u - 2\cos\frac{2j\pi}{m}\right)^2 = 2\cos mz - 2$$

Now we consider some paths in our surface. Let k be an odd number with $1 \leq k < m$. Let $z = \frac{k\pi}{m} + 2it$ for $t \in \mathbb{R}$. Thus each k determines a path in the u-plane. Let l be an integer with $0 \leq l < 2n$. The constant l will determine the lift of the curve in the u plane to X_0. Let $\gamma_{k,l} \subset X_0$ denote the path parametrized by $t \in \mathbb{R}$ and defined by

$$u = u(t) = 2\cos z = 2\cos(\frac{k\pi}{m} + 2it)$$
$$y = y(t) = 2^{\frac{1}{n}} e^{(\frac{1}{2n}+\frac{l}{n})i\pi}(-1)^{\frac{k-1}{2}} \cosh^{\frac{1}{n}} mt.$$

We have that both the right and left hand sides of the expression for X_0 equal $-4\cosh^2 mt$, so that γ_k does indeed lie on X_0. The curves $\gamma_{k,*}$ are pull backs of hyperbolas in u-plane which separate the point $u = 2\cos\frac{(k-1)\pi}{m}$ from $u = 2\cos\frac{(k+1)\pi}{m}$. Moreover, it can be seen that these curves never intersect, because the hyperbolas do not.

We will now work on simplifying ω_0 restricted to $\gamma_{k,l}$.

$$\omega_0\big|_{\gamma_{k,l}} = \frac{y(-2\sin z)\mathrm{d}z}{d} = \frac{y(-4\sin\frac{z}{2}\cos\frac{z}{2})\mathrm{d}z}{-4\sin\frac{mz}{2}\sin\frac{z}{2}} = \frac{y\cos\frac{z}{2}\mathrm{d}z}{\sin\frac{mz}{2}}.$$

We have that $\sin\frac{mz}{2} = \sin\frac{k\pi}{2}\cosh mt = (-1)^{\frac{k-1}{2}}\cosh mt$. The power of -1 cancels, thus

$$\omega_0\big|_{\gamma_{k,l}} = \frac{2^{\frac{1}{n}} e^{(\frac{1}{2n}+\frac{l}{n})i\pi}\cos\frac{z}{2}\mathrm{d}z}{\cosh^{1-\frac{1}{n}} mt} = \frac{2^{1+\frac{1}{n}} e^{(\frac{1}{2n}+\frac{l}{n})i\pi} i\cos(\frac{k\pi}{2m}+it)\mathrm{d}t}{\cosh^{1-\frac{1}{n}} mt} \tag{10}$$

The holonomy along the curve γ_k is given by

$$\int_{\gamma_{k,l}} \omega_0 = 2^{1+\frac{1}{n}} e^{(\frac{1}{2n}+\frac{l}{n})i\pi} i \int_{-\infty}^{\infty} \frac{\cos(\frac{k\pi}{2m}+it)}{\cosh^{1-\frac{1}{n}} mt}\,\mathrm{d}t$$

Note that the denominator in this integral is always real, and the numerator satisfies $\overline{\cos(\frac{k\pi}{2m}+it)} = \cos(\frac{k\pi}{2m}-it)$. Thus,

$$\int_{\gamma_{k,l}} \omega_0 = 2^{1+\frac{1}{n}} e^{(\frac{1}{2n}+\frac{l}{n})i\pi} i \int_0^{\infty} \frac{2\,\mathrm{Re}\big(\cos(\frac{k\pi}{2m}+it)\big)}{\cosh^{1-\frac{1}{n}} mt}\,\mathrm{d}t = 2^{2+\frac{1}{n}} e^{(\frac{1}{2n}+\frac{l}{n})i\pi} i\cos\frac{k\pi}{2m}\int_0^{\infty}\frac{\cosh t}{\cosh^{1-\frac{1}{n}} mt}\,\mathrm{d}t$$

Let C be the complex constant $2^{2+\frac{1}{n}} e^{\frac{i\pi}{2n}} i\int_0^{\infty}\frac{\cosh t}{\cosh^{1-\frac{1}{n}} mt}\,\mathrm{d}t$. Then we have

$$\int_{\gamma_{k,l}} \omega_0 = Ce^{\frac{li\pi}{n}}\cos\frac{k\pi}{2m}.$$

The points of the form $2\cos\frac{2j}{m}$ on X_0 are surrounded by the curves $\gamma_{2j-1,*}$ and $\gamma_{2j+1,*}$. Thus the holonomies of the curves surrounding lifts of the point $2\cos\frac{2j}{m}$ are the same as the holonomies of the semiregular 2n-gon

$$P_n(\cos\frac{(2j-1)\pi}{2m}, \sin\frac{(2j+1)\pi}{2m}) = P_n(\sin\frac{(m-2j+1)\pi}{2m}, \sin\frac{(m-2j-1)\pi}{2m}), \tag{11}$$

up to uniform scaling and rotation given by multiplication by C.

Now we must show that these arcs can be homotoped to geodesic segments. We begin by rewriting our formula for ω_0 that appears in equation 10. Let $c_l = 2^{1+\frac{1}{n}} e^{(\frac{1}{2n}+\frac{l}{n})i\pi} i$.

$$\omega_0\big|_{\gamma_{k,l}} = \frac{c_l\cos(\frac{k\pi}{2m}+it)\mathrm{d}t}{\cosh^{1-\frac{1}{n}} mt} = \frac{c_l\big((\cos(\frac{k\pi}{2m}) - i\sin(\frac{k\pi}{2m}))e^t + (\cos(\frac{k\pi}{2m}) + i\sin(\frac{k\pi}{2m}))e^{-t}\big)\ \mathrm{d}t}{\cosh^{1-\frac{1}{n}} mt}$$

Note that $\arg c_l$ is constant in terms of l, which points in the direction of the holonomy of $\gamma_{k,l}$. This formula shows that $|\arg\omega_0\big|_{\gamma_{k,l}}/c_l| < \frac{(m-k)\pi}{2m} < \frac{\pi}{2}$. Moreover, $\arg\omega_0\big|_{\gamma_{k,l}}/c_l$ is always decreasing in t. Thus, $\gamma_{k,l}$ always turns rightward as t increases. The component of

$(X_0, \omega_0) \smallsetminus \bigcup_{k,l} \gamma_{k,l}$ containing the point where $u = 2$ is bounded by the curves $\gamma_{1,l}$, which travel clockwise around the point where $u = 2$. Thus, this component must have a"flower shape," as in left side of figure 10. It follows that the curves $\gamma_{1,l}$ can be straightened to geodesics. Now assume the curves $\gamma_{k,l}$ have been straightened to geodesics. We will show that the curves $\gamma_{k+1,l}$ can also be straightened. The components $\gamma_{k,l}$ and $\gamma_{k+1,l}$ bound the two regions containing the two points where $u = 2\cos\frac{(k+1)\pi}{m}$. We may assume to have already straightened the $\gamma_{k,l}$. The curves $\gamma_{k+1,l}$ turn rightward as the they travel around the region in the clockwise direction. Hence, they look like the curved boundary edges of the region on the right side of figure 10. These curves can also be straightened to geodesics.

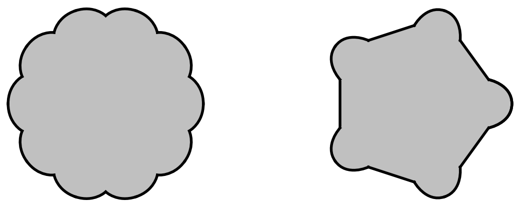

FIGURE 10. A caricature of the regions bounded by the curves $\gamma_{k,l}$ before straightening. Here $n = 5$.

The polygons given in equation 11, whose edges are the straightened versions of $\gamma_{k,l}$, are the same polygons used to build the surface $S'_{m,n}$, up to multiplication by C. Moreover, after indexing the polygons appropriately, we can see that these polygons are glued together in the same combinatorial way. □

The following is the second part of theorem 5.15 of [BM06].

Theorem 25 (Bouw-Möller). *Suppose m is even, n is odd and that n and m are relatively prime. The surface X_0 together with the differential ω_0 given below determines a translation surface with an affine automorphism group which is an (m, n, ∞) triangle group.*

$$X_0: \quad y^{2n} = (u-2)^n \prod_{j=1}^{m/2} \left(u - 2\cos\frac{(2j-1)\pi}{m}\right)^2$$

$$\omega_0 = \frac{y\ du}{(u-2)\prod_{j=1}^{m/2}(u - 2\cos\frac{(2j-1)\pi}{m})}$$

Proposition 26. *Up to scaling, there is an isometry from the translation surface determined by the pair (X_0, ω_0) in theorem 25 and one component of the surface $S'_{m,n}$ defined in subsection 1.4.*

Proof. The structure of X_0 is slightly different. The map $u : X_0 \to \hat{\mathbb{C}}$ is $2:1$ at each point $u = 2\cos\frac{(2j-1)\pi}{m}$, $n:1$ at $u = 2$ and $1:1$ at $u = \infty$. We will cut along curves which are lifts of hyperbolas in the u-plane that separating the points $u = 2\cos\frac{(2j-1)\pi}{m}$. We will also cut along n disjoint paths through the lifts of $u = 2$. The components of (X_0, ω_0) with these arcs removed all have the symmetries of a semiregular $2n$-gon.

Let $u = 2\cos z$. Then we have that

$$\prod_{j=1}^{m/2} \left(u - 2\cos\frac{(2j-1)\pi}{m}\right) = 2\cos\frac{mz}{2}.$$

For $k \in \{0, 1, \ldots, \frac{m-2}{2}\}$, consider the path in the u-coordinate determined by $z = z(t) = \frac{2k\pi}{m} + 2it$ with $t \in \mathbb{R}$. We will use $l \in \{0, 1, \ldots, 2n-1\}$ to denote the choice of a lift to X_0. We define the curve $\gamma_{k,l}$ by

$$u = u(t) = 2\cos(\frac{2k\pi}{m} + 2it)$$

$$y = y(t) = (-1)^k e^{\frac{l\pi i}{n}} 2^{1+\frac{1}{n}} i \sin(\frac{z}{2}) \cosh^{\frac{1}{n}} mt = e^{\frac{l\pi i}{n}} 2^{1+\frac{1}{n}} i \sin(\frac{j\pi}{m} + it) \cosh^{\frac{1}{n}} mt$$

This curve lies on X_0 because both the left and right side of the defining equation equal $-2^{2n+2}\cosh^2(mt)\sin^{2n}(\frac{z}{2})$. Moreover, these curves do not intersect, however when $k = 0$ we have provided two names for the same curve. For all but $k = 0$, we can see disjointness by looking at the u-plane. When $k = 0$, the curves pass through the branch point $u = 2$. However, the y-coordinate changes sign as $\gamma_{0,l}$ passes through the time when $u = 2$. In fact for $l < n$, $\gamma_{0,l} = \gamma_{0,l+n}$ with opposite orientation. In particular, the curves $\gamma_{0,*}$ are continuous through $u = 2$.

We will simplify the formula for ω_0 restricted to these curves.

$$\omega_0\big|_{\gamma_{k,l}} = \frac{y \, \mathrm{d}u}{-8\sin^2\frac{z}{2}\cos\frac{mz}{2}} = \frac{y(-2\sin z)\,\mathrm{d}z}{-8\sin^2\frac{z}{2}\cos\frac{mz}{2}\,\mathrm{d}z} = \frac{y\cos\frac{z}{2}\,\mathrm{d}z}{2\sin\frac{z}{2}\cos\frac{mz}{2}}$$

Now we can expand y and z. Note that $\cos\frac{mz}{2} = (-1)^k\cosh mt$

$$\omega_0\big|_{\gamma_{k,l}} = \frac{(-1)^k e^{\frac{l\pi i}{n}} 2^{1+\frac{1}{n}} i \cosh^{\frac{1}{n}}(mt)\cos\frac{z}{2}\ (2i)\,\mathrm{d}t}{2\cos\frac{mz}{2}} = \frac{-e^{\frac{l\pi i}{n}} 2^{1+\frac{1}{n}}\cos(\frac{k\pi}{m} + it)\,\mathrm{d}t}{\cosh^{1-\frac{1}{n}}(mt)}$$

We can now simplify the formula for the holonomy of $\gamma_{k,l}$.

$$\int_{\gamma_{k,l}} \omega_0 = -e^{\frac{l\pi i}{n}} 2^{1+\frac{1}{n}} \int_{-\infty}^{\infty} \frac{\cos(\frac{k\pi}{m} + it)}{\cosh^{1-\frac{1}{n}}(mt)}\,\mathrm{d}t$$

Let $C = -2^{2+\frac{1}{n}}\int_0^\infty \frac{\cosh t}{\cosh^{1-\frac{1}{n}}(mt)}\,\mathrm{d}t$. As in the previous proof, we have

$$\int_{\gamma_{k,l}} \omega_0 = -e^{\frac{l\pi i}{n}} 2^{2+\frac{1}{n}}\cos(\frac{k\pi}{m})\int_0^\infty \frac{\cosh t}{\cosh^{1-\frac{1}{n}}(mt)}\,\mathrm{d}t = Ce^{\frac{l\pi i}{n}}\cos(\frac{k\pi}{m})$$

The components of (X_0, ω_0) with $\{\gamma_{k,l}\}$ removed which contain $u = 2\cos\frac{(2j-1)\pi}{m}$ are bounded by curves of the form $\gamma_{j-1,*}$ and $\gamma_{j,*}$. The semiregular $2n$-gon

$$P_n(\cos\frac{(j-1)\pi}{m}, \cos\frac{j\pi}{m}) = P_n(\sin\frac{(m-2j+2)\pi}{2m}, \sin\frac{(m-2j)\pi}{2m})$$

has edges with the same holonomy up to uniform scaling and rotation by multiplication by C.

It remains to check that these edges can be straightened. The curves $\gamma_{0,l}$ are actually already geodesic. And a calculation shows that the remaining $\gamma_{k,l}$ are turning rightward. Again, an inductive argument shows that each $\gamma_{k,l}$ can be straightened. After re-indexing the polygons, they are glued in the same combinatorial way. □

6. Semiregular decompositions of infinite translation surfaces

Recall that in section 1.4, we defined surfaces $S'_{m,n}$ as a union of semiregular polygons and saw that $S'_{m,n}$ was the image of $S_{m,n}$ under an element of the affine group.

In this section, we list analogs of the definition of the surface $S'_{m,n}$ for the infinite translation surfaces $S'_{\mathcal{I},\mathcal{J}}$ defined in he previous subsection. We state these results without proof. These decompositions can be obtained by cutting along positive diagonals of horizontal rectangles and degenerate rectangles as in section 4.

We define the first component of $S'_{\mathcal{I},\mathcal{J}}$ as the union of $P(k)$ for $k \in \mathcal{V}(\mathcal{I}) \cup \{0\}$. Here, each $P(k)$ will be a semiregular polygon or an infinite sided analog. We follow the same gluing conventions as before. For k odd, we identify the even sides of $P(k)$ with the opposite side of $P(k+1)$, and identify the odd sides of $P(k)$ with the opposite side of $P(k-1)$. (We will specify the gluing in more detail when needed.) We will also define the element $M \in GL(2,\mathbb{R})$ which sends $S_{\mathcal{I},\mathcal{J}}$ to $S'_{\mathcal{I},\mathcal{J}}$. Because R swaps the two components of $S_{\mathcal{I},\mathcal{J}}$, the second component of $S'_{\mathcal{I},\mathcal{J}}$ can be attained by applying $M \circ R \circ M^{-1} \in GL(2,\mathbb{R})$ to the first component.

We first deal with the case when $\mathcal{J} = \mathcal{L}_n$ for $n \geq 2$. In all these cases, we have a natural decomposition of the first component of $S_{\mathcal{I},\mathcal{J}}$. into semiregular $2n$-gons, $P(k)$.

(1) Let $\mathcal{J} = \mathcal{L}_n$. We will break in to subcases for $\mathcal{I}$. However, in all these cases, we define $M = \begin{bmatrix} \csc \frac{\pi}{n} & -\cot \frac{\pi}{n} \\ 0 & 1 \end{bmatrix} \in GL(2,\mathbb{R})$. And thus, $M \circ R \circ M^{-1} = \begin{bmatrix} -\cos \frac{\pi}{n} & -\sin \frac{\pi}{n} \\ \sin \frac{\pi}{n} & -\cos \frac{\pi}{n} \end{bmatrix}$.
   (a) Assume $\mathcal{I} = \mathcal{L}_{\mathbb{Z}}$. In this case, set $P(k) = P_n(1,1)$ for $k \in \mathbb{Z}$.
   (b) Assume $\mathcal{I} = \mathcal{L}_{\mathbb{N}}$ and that n is odd. In this case, set $P(k) = P_n(k+1,k)$ for $k \in \mathbb{N} \cup \{0\}$.
   (c) Assume $\mathcal{I} = \mathcal{L}_{\mathbb{N}}$ and that n is even. In this case, set $P(k) = P_n(k,k+1)$ for $k \in \mathbb{N} \cup \{0\}$ even, and set $P(k) = P_n(k+1,k)$ if $k \in \mathbb{N}$ is odd.

In order to cover the other cases, we need infinite analogs of a semiregular polygon. We first handle the case $\mathcal{J} = \mathcal{L}_{\mathbb{N}}$. For $i \in \mathbb{Z}$ and $a, b \geq 0$ not both zero, define the vectors

$$\mathbf{v}_i^+ = \begin{cases} (a, ai) & \text{if } i \text{ is even} \\ (b, bi) & \text{if } i \text{ is odd} \end{cases} \quad \text{and} \quad \mathbf{v}_i^- = \begin{cases} (a, -ai) & \text{if } i \text{ is even} \\ (b, -bi) & \text{if } i \text{ is odd} \end{cases}.$$

Define the *semiregular polygonal parabola* $P^+_{\mathbb{N}}(a,b)$ to be the set of points who lie above the polygonal parabola formed by translating the vectors v_i^+ so that the endpoint of v_i^+ aligns with the starting point of v_{i+1}^+ for all k. (The vertices of this polygonal parabola lie on an upward pointed parabola.) Similarly, define the polygonal parabola $P^-_{\mathbb{N}}(a,b)$ to be the set of points below the polygonal parabola formed by translating the vectors v_i^- so that the endpoint of v_i^- aligns with the starting point of v_{i+1}^+ for all k. Note that the even sides of $P^+_{\mathbb{N}}(a,b)$ can be glued to the even sides of $P^-_{\mathbb{N}}(a,c)$ in a unique way by translation. Also, The odd sides of $P^+_{\mathbb{N}}(a,b)$ can be glued to the odd sides of $P^-_{\mathbb{N}}(c,b)$.

(4) Let $\mathcal{J} = \mathcal{L}_{\mathbb{N}}$ and $\mathcal{I} = \mathcal{L}_m$ for $m \geq 2$, $\mathcal{I} = \mathcal{L}_{\mathbb{N}}$, or $\mathcal{I} = \mathcal{L}_{\mathbb{Z}}$. For $k \in \mathcal{V}(\mathcal{I}) \cup \{0\}$, set $P(k) = P^+_{\mathbb{N}}(f_{\mathcal{I}}(k+1), f_{\mathcal{I}}(k))$ if k is even and $P(k) = P^-_{\mathbb{N}}(f_{\mathcal{I}}(k), f_{\mathcal{I}}(k+1))$ if k is odd. Here $f_{\mathcal{I}}$ is the critical eigenfunction given in definition 20. (We treat $f_{\mathcal{I}}(k) = 0$ if $k \notin \mathcal{V}(\mathcal{I})$.) Define $M = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix} \in GL(2,\mathbb{R})$. Then $M \circ R \circ M^{-1} = \begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix}$.

An example is depicted in figure 11.

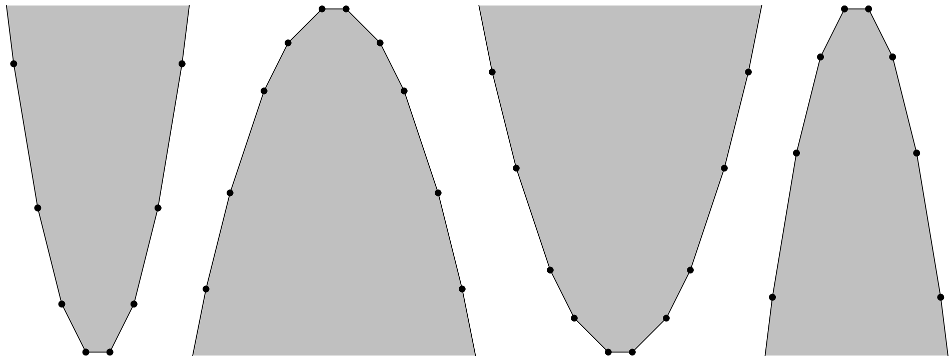

FIGURE 11. The first component of $S'_{\mathcal{I},\mathcal{J}}$ with $\mathcal{J} = \mathcal{L}_{\mathbb{N}}$ and $\mathcal{I} = \mathcal{L}_4$. These are the polygonal parabolas $P(0) = P^+_{\mathbb{N}}(\frac{\sqrt{2}}{2}, 0)$, $P(1) = P^-_{\mathbb{N}}(\frac{\sqrt{2}}{2}, 1)$, $P(2) = P^+_{\mathbb{N}}(\frac{\sqrt{2}}{2}, 1)$ and $P(3) = P^-_{\mathbb{N}}(\frac{\sqrt{2}}{2}, 0)$ from left to right.

Now we will cover the case $\mathcal{J} = \mathcal{L}_{\mathbb{Z}}$. For $a, b \geq 0$ not both zero and $k \in \mathbb{Z}$, define the segments

$$e_i^{\mathrm{R}} = \begin{cases} \overline{\big(a, \frac{(i-1)a}{2} + \frac{ib}{2}\big)\big(a, \frac{(i+1)a}{2} + \frac{ib}{2}\big)} & \text{if } i \text{ is even} \\ \overline{\big(a, \frac{ia}{2} + \frac{(i-1)b}{2}\big)\big(a, \frac{ia}{2} + \frac{(i+1)b}{2}\big)} & \text{if } i \text{ is odd} \end{cases}$$

$$e_i^{\mathrm{L}} = \begin{cases} \overline{\big(-b, \frac{-(i-1)a}{2} + \frac{-ib}{2}\big)\big(-b, \frac{-(i+1)a}{2} + \frac{-ib}{2}\big)} & \text{if } i \text{ is even} \\ \overline{\big(-b, \frac{-ia}{2} + \frac{-(i-1)b}{2}\big)\big(-b, \frac{-ia}{2} + \frac{-(i+1)b}{2}\big)} & \text{if } i \text{ is odd.} \end{cases}$$

Define the semiregular strip $P_{\mathbb{Z}}(a,b)$ to be the strip $P_{\mathbb{Z}}(a,b) = \{(x,y) \in \mathbb{R}^2 \ : \ -b \leq x \leq a\}$, with edges given by e_i^{left} and e_i^{right}.

(5) Let $\mathcal{J} = \mathcal{L}_{\mathbb{Z}}$ and $\mathcal{I} = \mathcal{L}_m$ for $m \geq 2$, $\mathcal{I} = \mathcal{L}_{\mathbb{N}}$, or $\mathcal{I} = \mathcal{L}_{\mathbb{Z}}$. For $k \in \mathcal{V}(\mathcal{I}) \cup \{0\}$, set $P(k) = P_{\mathbb{Z}}(f_{\mathcal{I}}(k+1), f_{\mathcal{I}}(k))$ if k is even and $P(k) = P_{\mathbb{Z}}(f_{\mathcal{I}}(k), f_{\mathcal{I}}(k+1))$ if k is odd. (We treat $f_{\mathcal{I}}(k) = 0$ if $k \notin \mathcal{V}(\mathcal{I})$.) Again we define $M = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix} \in GL(2, \mathbb{R})$ and $M \circ R \circ M^{-1} = \begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix}$.

Let us describe the gluing of these strips in detail. For k odd and i odd, we glue the sides e_i^{R} of $P(k)$ to side e_i^{L} of $P(k-1)$, and we glue side e_i^{L} of $P(k)$ to side e_i^{R} of $P(k-1)$. For k odd and i even, we glue the sides e_i^{R} of $P(k)$ to side e_i^{L} of $P(k+1)$, and we glue side e_i^{L} of $P(k)$ to side e_i^{R} of $P(k+1)$. See figure 12 for an example.

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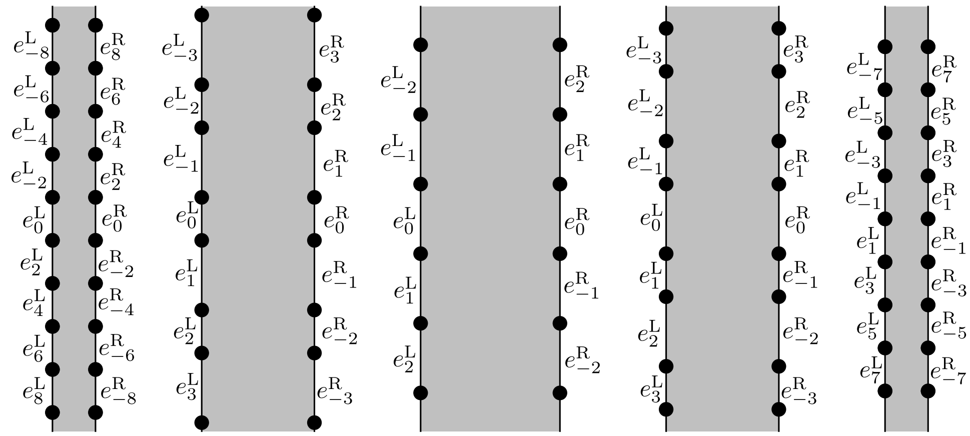

FIGURE 12. The first component of $S'_{\mathcal{I},\mathcal{J}}$ with $\mathcal{J} = \mathcal{L}_{\mathbb{Z}}$ and $\mathcal{I} = \mathcal{L}_5$. From left to right, these are the semiregular strips $P(0) = P_{\mathbb{Z}}(\sin\frac{\pi}{5}, 0)$, $P(1) = P_{\mathbb{Z}}(\sin\frac{\pi}{5}, \sin\frac{2\pi}{5})$, $P(2) = P_{\mathbb{Z}}(\sin\frac{3\pi}{5}, \sin\frac{2\pi}{5})$, $P(3) = P_{\mathbb{Z}}(\sin\frac{3\pi}{5}, \sin\frac{4\pi}{5})$ and $P(4) = P_{\mathbb{Z}}(0, \sin\frac{4\pi}{5})$.

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