

# IMMERSIONS AND THE SPACE OF ALL TRANSLATION STRUCTURES

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ABSTRACT. A translation structure on a surface is an atlas of charts to the plane so that the transition functions are translations. We allow our surfaces to be non-compact and infinite genus. We endow the space of all pointed surfaces equipped with a translation structure with a topology, which we call the immersive topology because it is related to the manner in which disks can be immersed into such a surface. We prove that a number of operations typically done to translation surfaces are continuous with respect to the topology. We show that the topology is Hausdorff, and that the collection of surfaces with a fixed lower bound on the injectivity radius at the basepoint is compact.

## 1. INTRODUCTION

A *translation structure* on a surface is an atlas of charts to the plane where the transition functions are translations. There is a natural notion of when two such structures are isomorphic (as we explain in §2), and we use the term *translation surface* to mean an isomorphism class of translation structures. A *pointed translation surface* is a connected translation surface together with a choice of a basepoint.

Translation surfaces are relatively simple geometric objects. Despite their simplicity, there are natural geometric and dynamical questions about these surfaces whose difficulty varies greatly based on the surface chosen.

The primary goal of this article is to topologize the collection of *all* pointed translation surfaces. That is, the space will include translation surfaces of any topological type admitting a translation structure. The topology is designed to allow sequences of infinite genus surfaces to converge, and to allow finite genus surfaces to limit on surfaces of infinite genus. The topology also makes it easy to pass geometric information back and forth from the limiting surface to the approximates.

As motivation, we note that translation surfaces come in great variety, and highlight a few examples of interest:

- The only closed surfaces admitting translation structures are tori.
- Oriented surfaces,  $S_{g,n}$ , of genus  $g \geq 2$  with  $n \geq 1$  punctures always admit translation structures whose metric completions are homeomorphic to closed surfaces of genus  $g$ . Such metric completions introduce cone singularities with cone angles in  $2\pi\mathbb{Z}$ .
- A natural unfolding construction associates a translation surface to every polygonal billiard table [ZK75]. If the polygon is *irrational* (has angles which are irrational multiples of  $\pi$ ), then the translation surface has infinite area and infinite genus.

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These examples are listed in order of our understanding. The space of translation structures on tori is classically identified with the space of lattices in the plane,  $GL(2, \mathbb{R})/GL(2, \mathbb{Z})$ . The completions of genus  $g \geq 2$  surfaces are traditionally just called translation surfaces, but we will call them *finite genus translation surfaces*. The moduli space of such translation surfaces of genus  $g$  can be identified with a vector bundle (with the zero section removed) over the moduli spaces of Riemann surfaces of genus  $g$ . These surfaces have been actively studied for the past 35 years. We refer the reader to the survey articles [MT02] and [Zor06] for an introduction to the subject. At the other extreme, billiards in irrational polygons are not well understood at all. Basic questions such as the existence of periodic billiard orbits remain unanswered. (Existence of periodic billiard paths is equivalent to the existence of closed geodesics in the associated translation surface.) We refer the reader to [Tab05] for background on polygonal billiards.

The space of all translation surfaces is attractive because it contains all the surfaces mentioned in the above. In particular, it is natural to wonder which surfaces in this space can be best understood (from various viewpoints). Perhaps a reasonable expectation is that an infinite genus limit of the best understood finite genus translation surfaces should be easier to understand than some high genus translation surfaces.

Indeed, work from the past few years on the geometry and dynamics of infinite genus translation surfaces seems to suggest that there are many such surfaces which can be well understood. There has been widespread recent interest in the study of abelian covers of closed finite genus translation surfaces. Many such articles studied covers of *lattice surfaces* (maximally symmetric finite genus translation surfaces in the sense of Veech [Vee89]): [HS10], [HHW13], [Tro10], [DHL11], [HLT11], [Sch11], [HW13], and [Del13]. A few papers have studied  $\mathbb{Z}$ -covers of less symmetric finite genus translation surfaces including [HW12], [FU11] and [FU12], [RT13], and [RT12]. A couple of articles have studied surfaces arising from limiting procedures such as [Bow13] and [Hoo13b]. And a number of articles have studied surfaces with no apparent connection to the finite genus case including [PSV11], [BV13], [Tre12] and [Hoo10].

Aside from the perceived practical benefits for defining the topology, it is hoped that by formally placing all translation surfaces in the same space, we will provide a context for better understanding progress in the subject.

We briefly discuss the layout of this paper. In §2, we give a set-theoretic description of the moduli space of all translation structures. We also introduce the canonical bundle of translation surfaces over this moduli space. In §3, we introduce the idea of immersing one subset of a pointed translation surface into another pointed translation surface. We use this idea to place a topology on the space of pointed translation surfaces. In §4, we describe our main results for this topology. We are interested in understanding this topological space and showing that a number of natural operations done on pointed translation surfaces are continuous. In §5, we explain our approach to the proofs of the main results. We will utilize our understanding of translation structures on the disk developed in [Hoo13a]. In §6, we outline the remainder of the paper and describe where the main results are proved.

## 2. TRANSLATION STRUCTURES

**2.1. Definition of translation structure.** We take a view of translation structures following the more general idea of a  $(G, X)$ -structure. See [Thu97] for an introduction to these ideas.

In this paper, we use the term *surface* to mean a connected oriented 2-manifold  $X$  without boundary (which may or may not be closed). An *atlas of charts* from a surface  $X$  to  $\mathbb{R}^2$  is a collection of orientation preserving local homeomorphisms from open subsets of  $X$  to  $\mathbb{R}^2$  so that the collection of domains of these maps cover  $X$ . We denote the choice of an atlas by a set of pairs consisting of the domain and the map,  $\mathcal{A} = \{(U_j, \phi_j)\}$ . Such an atlas  $\mathcal{A}$  is a *translation atlas* if for every choice of  $i$  and  $j$  so that  $U_i \cap U_j \neq \emptyset$ , the associated *transition function*,

$$\phi_i \circ \phi_j^{-1}|_{\phi_j(U_i \cap U_j)} : \phi_j(U_i \cap U_j) \rightarrow \mathbb{R}^2$$

is locally a restriction of a translation. A translation atlas on  $X$  is *maximal* if it is not properly contained in any translation atlas on  $X$ . A *translation structure* is a pair  $(X, \mathcal{A})$ , where  $X$  is a surface and  $\mathcal{A}$  is a maximal translation atlas on  $X$ .

Because transition functions are translations, a translation atlas determines a metric on  $X$  via pullback. It also determines a canonical trivialization of the tangent bundle of  $X$ ,  $TX \cong X \times \mathbb{R}^2$ , by pulling back the standard trivialization of the unit tangent bundle of  $\mathbb{R}^2$ ,  $T\mathbb{R}^2 \cong \mathbb{R}^2 \times \mathbb{R}^2$ .

**2.2. Translation isomorphisms and moduli space.** Let  $(X, \mathcal{A})$  and  $(Y, \mathcal{B})$  be translation structures, and let  $\mathcal{A} = \{(U_i, \phi_i)\}$  and  $\mathcal{B} = \{(V_j, \psi_j)\}$ . A homeomorphism  $h : X \rightarrow Y$  is a *translation isomorphism* from the structure  $\mathcal{A}$  to the structure  $\mathcal{B}$  if

$$\mathcal{B} = \left\{ (h(U_i), \phi_i \circ h^{-1} : h(U_i) \rightarrow \mathbb{R}^2) \right\}.$$

That is, we require that the push forward of the atlas  $\mathcal{A}$  under  $h$  is the same as the atlas  $\mathcal{B}$ . If such an isomorphism exists we call  $(X, \mathcal{A})$  and  $(Y, \mathcal{B})$  *translation isomorphic*.

In order to put a topology on the space of translation structures, we will need to add a basepoint. A *pointed surface* is a pair  $(X, x_0)$  consisting of a connected surface  $X$  and a *basepoint*  $x_0 \in X$ . A *pointed translation structure* is a triple  $(X, x_0, \mathcal{A})$  where  $(X, x_0)$  is a pointed surface and  $(X, \mathcal{A})$  is a translation structure. An isomorphism between the pointed translation structures  $(X, x_0, \mathcal{A})$  and  $(Y, y_0, \mathcal{B})$  is a translation isomorphism  $h : X \rightarrow Y$  which respects the basepoints:  $h(x_0) = y_0$ . If such an isomorphism exists, we call the structures *isomorphic*. We write  $[X, x_0, \mathcal{A}]$  to denote the isomorphism class of the pointed translation structure  $(X, x_0, \mathcal{A})$ .

The *moduli space*  $\mathcal{M}$  of all (pointed) translation structures is the collection of all isomorphism classes of pointed translation structures. This moduli space is naturally partitioned according to the homeomorphism type of the underlying surface. Given  $(X, x_0)$ , we let  $\mathcal{M}(X, x_0)$  denote the collection of all isomorphism classes of the form  $[X, x_0, \mathcal{A}]$ . Then we have

$$\mathcal{M} = \bigcup_{(X, x_0) \in \mathcal{S}} \mathcal{M}(X, x_0),$$

where  $\mathcal{S}$  is a collection of pointed surfaces with one representative from each homeomorphism class of surfaces.

**2.3. Bundle structure.** We will describe a canonical (set-theoretic) surface bundle over  $\mathcal{M}$  so that the fiber over a point in  $\mathcal{M}(X, x_0)$  is isomorphic to  $X$ .

Consider the collection of all pointed translation surfaces with an additional point selected,

$$E = \{(X, x_0, \mathcal{A}; x) : (X, x_0, \mathcal{A}) \text{ is a translation structure and } x \in X\}.$$

A homeomorphism  $h : X \rightarrow X$  is a *two pointed isomorphism* from  $(X, x_0, \mathcal{A}; x) \in E$  to  $(Y, y_0, \mathcal{B}; y) \in E$  if it is an isomorphism from  $(X, x_0, \mathcal{A})$  to  $(Y, y_0, \mathcal{B})$  and satisfies  $h(x) = y$ . We write  $[X, x_0, \mathcal{A}; x]$  to denote the two pointed isomorphism class of  $(X, x_0, \mathcal{A}; x)$ . We define the *total space  $\mathcal{E}$  of translation surfaces* to be the collection of all two pointed isomorphism classes.

Note that there is a canonical projection

$$\pi : \mathcal{E} \rightarrow \mathcal{M}; \quad [X, x_0, \mathcal{A}; x] \mapsto [X, x_0, \mathcal{A}].$$

Moreover, the fibers of this projection are endowed with a natural translation surface structure. To see this, choose an  $[X, x_0, \mathcal{A}] \in \mathcal{M}$  and a translation structure  $(X, x_0, \mathcal{A})$  from the isomorphism class  $[X, x_0, \mathcal{A}]$ . Let  $\mathcal{A} = \{(U_i, \phi_i)\}$ , and let  $S = \pi^{-1}([X, x_0, \mathcal{A}]) \subset \mathcal{E}$  be the fiber. Observe that associated to choice of  $(X, x_0, \mathcal{A}) \in [X, x_0, \mathcal{A}]$ , we have a canonical bijection

$$\gamma : X \rightarrow S; \quad x \mapsto [X, x_0, \mathcal{A}; x].$$

We can use  $\gamma$  to push the topology from  $X$  onto  $S$ . This makes  $S$  a surface. Furthermore, this topology is independent of our choice. In addition, we get a translation structure on  $S$  from the atlas

$$\{(\gamma(U_i), \phi_i \circ \gamma^{-1})\}.$$

Again, this atlas is independent of the choice of translation structure from the isomorphism class  $[X, x_0, \mathcal{A}; x]$ . In summary, we have endowed each fiber  $S = \pi^{-1}([X, x_0, \mathcal{A}])$  with a translation structure which is isomorphic to each of the translation structures in the equivalence class  $[X, x_0, \mathcal{A}; x]$ .

**Convention 1.** A *translation surface* is a fiber of the projection  $\pi : \mathcal{E} \rightarrow \mathcal{M}$  endowed with the topology of a surface and a translation structure as described above. We will denote a translation surface by capitol letters such as  $R, S$  or  $T$ , and points on the translation surface by lower case letters such as  $r, s$ , or  $t$ . A translation surface  $S = \pi^{-1}([X, x_0, \mathcal{A}])$  has a canonical basepoint,  $o_S = [X, x_0, \mathcal{A}; x_0]$ . We will identify each point in  $\mathcal{M}$  with the translation surface above the point. This allows us to write  $S \in \mathcal{M}$  without reference to equivalence classes. Note that  $S \subset \mathcal{E}$ , so points in the translation surface  $S$  also belong to  $\mathcal{E}$ . But, we will only rarely want to refer to a point  $s \in \mathcal{E}$  without referring to the translation surface  $S = \pi(s)$  on which the point lies. Frequently, we will redundantly write points of  $\mathcal{E}$  as pairs  $(S, s) \in \mathcal{E}$  where  $S = \pi(s)$ .

### 3. IMMERSIONS AND TOPOLOGIES

**3.1. Definition of immersion.** Let  $S$  be a translation surface. We let  $\text{PC}(S)$  denote the collection of all path-connected subsets of  $S$  that contain the basepoint  $o_S$ .

Let  $S$  and  $T$  be a translation surfaces. Let  $A \in \text{PC}(S)$  and let  $B \subset T$  be an arbitrary subset. An *immersion* of  $A$  into  $B$  is a continuous map  $\iota : A \rightarrow B$  which respects the basepoint and the translation structures. That is, we require:

- $\iota(o_S) = o_T$ .
- For all  $s \in A$ , there is a choice chart  $(U, \phi)$  in the maximal translation atlas of  $S$ , with  $s \in U$ , a chart  $(V, \psi)$  in the maximal translation atlas of  $T$ , and a vector  $\mathbf{v} \in \mathbb{R}^2$  so that  $\iota(U \cap A) \subset V$  and

$$\psi \circ \iota(s') = \mathbf{v} + \phi(s') \quad \text{for all } s' \in U \cap A.$$

If there is an immersion of  $A$  into  $B$ , we say  $A$  *immerses* in  $B$  and write  $A \rightsquigarrow B$ . We write  $\exists \iota : A \rightsquigarrow B$  to represent the statement “there is an immersion,  $\iota$  from  $A$  into  $B$ .” If there is no such immersion, we write  $A \not\rightsquigarrow B$ .

**Proposition 2.** *If there is an immersion  $\iota : A \rightsquigarrow B$ , then it is unique.*

*Proof.* Since  $A$  is path-connected and our immersion must respect the basepoint, the immersion is determined by analytic continuation.  $\square$

An injective immersion is called an *embedding*. We denote the statement “there exists an embedding  $e$  of  $A$  into  $B$ ” by  $\exists e : A \hookrightarrow B$ . We follow the same notational scheme as for immersions.

**Corollary 3.** *Suppose  $A \in PC(S)$  and  $B \in PC(T)$ . If  $A \rightsquigarrow B$  and  $B \rightsquigarrow A$ , then both immersions are embeddings, and the two embeddings are inverses of one another.*

*Proof.* Both the identity map on  $A$  and the composition  $A \rightsquigarrow B \rightsquigarrow A$  are immersions. They are identical, because of the uniqueness of immersions. The conclusion follows.  $\square$

If the statement of the corollary is satisfied for  $A$  and  $B$ , then we say these are *isomorphic subsets* of translation surfaces. Isomorphic subsets are indistinguishable from the point of view of immersions and embeddings:

**Corollary 4.** *Let  $A \in PC(S)$  and  $B \subset T$ . The truth of the statements  $A \rightsquigarrow B$  and  $A \hookrightarrow B$  do not depend on the choice of representative of  $A$  from its isomorphism class. If  $B \in PC(T)$ , the same holds for the choice of  $B$  from its isomorphism class.*

**3.2. The immersive topology on  $\mathcal{M}$ .** Let PC denote the collection of all path connected subsets of translation surfaces that contain the basepoint. An *open disk* is a set in PC that is homeomorphic to an open disk. A *closed disk* is a set in PC that is homeomorphic to a closed disk and contains the basepoint in its interior. We denote the collection of all closed disks in a translation surface  $S$  by  $\overline{\text{Disk}}(S)$ , and the set of all open disks in  $S$  by  $\text{Disk}(S)$ . However, we will frequently refer to open and closed disks without referring to the surface which contains them.

The *immersive topology* on  $\mathcal{M}$  is the coarsest topology so that the following list of sets are all open:

- Sets of the form  $\mathcal{M}_{\rightsquigarrow}(D) = \{S \in \mathcal{M} : D \rightsquigarrow S\}$ , where  $D$  is a closed disk.
- Sets of the form  $\mathcal{M}_{\not\rightsquigarrow}(U) = \{S \in \mathcal{M} : U \not\rightsquigarrow S\}$ , where  $U$  is an open disk.
- Sets of the form

$$\mathcal{M}_+(D, U) = \{S \in \mathcal{M} : \exists \iota : D \rightsquigarrow S \text{ and } o_S \in \iota(U)\},$$

where  $D$  is a closed disk and  $U$  is an open subset of the interior,  $D^\circ$ .

- Sets of the form

$$\mathcal{M}_-(D, K) = \{S \in \mathcal{M} : \exists \iota : D \rightsquigarrow S \text{ and } o_S \notin \iota(K)\},$$

where  $D$  is a closed disk and  $K \subset D$  is closed.

We will find the following results useful:

**Theorem 5** (Embedding Theorem). *If  $D$  is a closed disk, then  $\mathcal{M}_{\hookrightarrow}(D)$  is open.*

**Theorem 6** (Disjointness Theorem). *If  $D \in PC$  is a closed disk, and  $K_1$  and  $K_2$  are disjoint closed subsets of  $D$ , then the following set is open in  $\mathcal{M}$ :*

$$\mathcal{M}_\emptyset(D; K_1, K_2) = \{S \in \mathcal{M} : \exists \iota : D \rightsquigarrow S \text{ and } \iota(K_1) \cap \iota(K_2) = \emptyset\}.$$

**3.3. The immersive topology on  $\mathcal{E}$ .** We will define a topology on  $\mathcal{E}$  which builds off of the topology we defined on  $\mathcal{M}$  above. The *immersive topology* on  $\mathcal{E}$  is the coarsest topology so that the projection  $\pi : \mathcal{E} \rightarrow \mathcal{M}$  is continuous and so that the set

$$\mathcal{E}_+(D, U) = \{(S, s) \in \mathcal{E} : \exists \iota : D \rightsquigarrow S \text{ and } s \in \iota(U)\}$$

is open whenever  $D$  is a closed disk and  $U$  is an open subset of its interior  $D^\circ$ .

Later in the paper, we will prove the following:

**Proposition 7.** *If  $D$  is a closed disk and  $K \subset D$  is closed, then the following set is open:*

$$\mathcal{E}_-(D, K) = \{(S, s) \in \mathcal{E} : \exists \iota : D \rightsquigarrow S \text{ and } s \notin \iota(K)\}.$$

#### 4. MAIN RESULTS

The following result guarantees that the spaces  $\mathcal{M}$  and  $\mathcal{E}$  are fairly reasonable topological spaces. In particular, limits are unique.

**Theorem 8.** *The immersive topologies on  $\mathcal{M}$  and  $\mathcal{E}$  are second countable and Hausdorff.*

We find it useful to observe that immersions of open disks vary continuously in both the domain of the immersion and the choice of the target:

**Proposition 9** (Joint continuity of immersions). *Let  $U$  be an open disk, and let  $\mathcal{I}(U) \subset \mathcal{M}$  denote those  $S \in \mathcal{M}$  so that  $U \rightsquigarrow S$ . For  $S \in \mathcal{I}(U)$ , let  $\iota_S : U \rightsquigarrow S$  be the associated immersion. Then, the following map is continuous:*

$$I_U : \mathcal{I}(U) \times U \rightarrow \mathcal{E}; \quad (S, u) \mapsto \iota_S(u)$$

Since our surfaces are pointed, it is reasonable to ask what happens when the basepoint is moved. There are two important maps related to this idea. First, if  $S \in \mathcal{M}$  and  $s \in S$ , then we define  $\text{BC}(S, s) = S^s \in \mathcal{M}$  to be the translation surface which is isomorphic to  $S$  with the basepoint relocated to  $s$ . This defines the *basepoint changing map*  $\text{BC} : \mathcal{E} \rightarrow \mathcal{M}$ .

**Theorem 10.** *The basepoint changing map  $\text{BC} : (S, s) \mapsto S^s$  is continuous.*

In addition to the basepoint changing map, there is a *basepoint changing isomorphism*  $\beta_s : S \rightarrow S^s$ . This is the translation isomorphism which sends  $s \in S$  to the basepoint of  $S^s$ .

**Theorem 11.** *Consider the basepoint changing isomorphism  $s' \mapsto \beta_s(s')$  and the inverse basepoint changing isomorphism, which sends  $t \in S^s$  to  $\beta_s^{-1}(t)$ . Both maps are jointly continuous in both  $s$  and the given domain ( $S$  and  $S^s$ , respectively).*

There is a standard action of elements  $A \in GL(2, \mathbb{R})$  on a translation structures on a surface  $X$ . If  $\mathcal{A} = \{(U_j, \phi_j) : j \in \mathcal{J}\}$  is an atlas determining a translation structure on  $X$ , then the image of this structure under  $A$  is given by

$$A(\mathcal{A}) = \{(U_j, A \circ \phi_j) : j \in \mathcal{J}\}.$$

The fact that  $A(\mathcal{A})$  is also a translation structure on  $X$  follows from the observation that the group of translations of the plane is a normal subgroup of the affine group of the plane, which also contains  $GL(2, \mathbb{R})$ . The  $GL(2, \mathbb{R})$  action sends isomorphic translation structures to isomorphic translation structures, and thus induces an action on  $\mathcal{M}$ . Furthermore, there is a natural action of each  $A \in GL(2, \mathbb{R})$  on  $\mathcal{E}$  given by

$$A : \mathcal{E} \rightarrow \mathcal{E}; \quad [X, x_0, \mathcal{A}; x] \mapsto [X, x_0, A(\mathcal{A}); x].$$

Here, we are choosing a representative  $(X, x_0, \mathcal{A}; x)$  of  $[X, x_0, \mathcal{A}; x] \in \mathcal{E}$ , but the image is independent of this choice. Observe that this action restricted to a fiber of  $\pi : \mathcal{E} \rightarrow \mathcal{M}$  restricts to a homeomorphism between each translation surface and the image surface. That is, if  $S \in \mathcal{M}$  is a translation surface, then  $A|_S : S \rightarrow A(S)$  is a homeomorphism. The action is natural in the sense that for any chart  $(U, \phi)$  for the translation structure on  $S$ , there is a chart  $(V, \psi)$  for the translation structure on  $A(S)$  so that  $A(U) = V$  and the following diagram commutes:

$$(1) \quad \begin{array}{ccc} U & \xrightarrow{A|_S} & V \\ \downarrow \phi & & \downarrow \psi \\ \mathbb{R}^2 & \xrightarrow{A} & \mathbb{R}^2 \end{array}$$

**Theorem 12** (Continuity of affine actions). *The actions of  $GL(2, \mathbb{R})$  on  $\mathcal{M}$  and  $\mathcal{E}$  are continuous.*

An *affine automorphism* of a translation surface  $S$  is a homeomorphism  $S \rightarrow S$  which respects the affine structure underlying the translation structure on  $S$ . More concretely, a homeomorphism  $h : S \rightarrow S$  is an affine automorphism if there is an  $A \in GL(2, \mathbb{R})$  so that for each chart  $(U, \phi)$  in the maximal translation atlas on  $S$ , the pair  $(h(U), A \circ \phi)$  is also a chart in this atlas. We call  $A \in GL(2, \mathbb{R})$  the *derivative* of the affine automorphism  $h$ .

Note that we do not require an affine automorphism  $h : S \rightarrow S$  to respect the basepoint. If  $s = h^{-1}(o_s)$  is the preimage of the basepoint, then  $A(S^s) = S$  and  $h$  is given by the composition

$$S \xrightarrow{\beta_s} S^s \xrightarrow{A} S,$$

where  $\beta_s$  is the basepoint changing isomorphism. By continuity of the  $GL(2, \mathbb{R})$  actions and joint continuity of  $\beta_s$  and  $\beta_s^{-1}$ , we have obtained the following:

**Corollary 13** (Convergence of affine automorphisms). *Let  $\langle S_n \in \mathcal{M} \rangle$  be a sequence of translation surfaces converging to a surface  $S \in \mathcal{M}$ . Suppose that each  $S_n$  admits an affine homeomorphism  $h_n$  with derivative  $A_n$ . Further suppose that  $\langle A_n \rangle$  converges to some  $A \in GL(2, \mathbb{R})$ , and that  $s_n = h_n^{-1}(o_{S_n})$  converges to some limit point  $s \in S$ . Then, there is an affine homeomorphism  $h : S \rightarrow S$  with derivative  $A$  so that  $h(s) = o_S$ . Moreover, for any sequence  $\langle t_n \in S_n \rangle$  tending to  $t \in S$ , the sequence  $\langle h_n^k(t_n) \rangle$  converges to  $h^k(t)$  for all  $k \in \mathbb{Z}$ .*

Finally, we will prove that the collection of surfaces containing an open disk is compact:

**Theorem 14** (Compactness). *Let  $U$  be an open disk in a translation surface containing the basepoint. The following set is compact:*

$$\mathcal{M} \setminus \mathcal{M}_{\nrightarrow}(U) = \{S \in \mathcal{M} : U \hookrightarrow S\}.$$

## 5. GENERAL APPROACH

The approach of this paper is to use results already proved in [Hoo13a] about the immersive topology on translation structures on disks. In this section, we explain some results which makes this approach work.

We begin by describing the philosophy of the approach. Each translation surface  $S \in \mathcal{M}$ , has a universal cover  $\tilde{S}$  which inherits a translation structure by pulling back the structure along the covering map  $p_S : \tilde{S} \rightarrow S$ . Let  $(\Delta, x_0)$  be the pointed disk and let  $\tilde{\mathcal{M}} = \mathcal{M}(\Delta, x_0) \subset$

$\mathcal{M}$  be the collection of all translation structures on the disk. Thus,  $\tilde{S}$  lies in  $\tilde{\mathcal{M}}$ , and we can recover  $S$  as the quotient of  $\tilde{S}$  modulo the deck group of the cover. So, another way to think of a point in  $\mathcal{M}$  is as a choice of a point in  $\tilde{\mathcal{M}}$  together with a discrete group of translation automorphisms of  $\tilde{\mathcal{M}}$ . This approach brings to mind the point of view of the universal family of curves over moduli space. See [Zvo12] for background.

A third viewpoint on the moduli space  $\mathcal{M}$  is that we can associate a point  $S \in \mathcal{M}$  to its universal cover  $\tilde{S} \in \tilde{\mathcal{M}}$  and the preimages of the basepoint under the covering map,  $p_S^{-1}(o_S)$ . The set  $p_S^{-1}(o_S)$  is a discrete subgroup of  $\tilde{S}$  with the property that  $\text{BC}(\tilde{S}, \tilde{s}) = \tilde{S}$  for each  $\tilde{s} \in p_S^{-1}(o_S)$ . We think of  $p_S^{-1}(o_S)$  as a subset of  $\tilde{\mathcal{E}} \subset \mathcal{E}$ , the canonical disk bundle over  $\tilde{\mathcal{M}}$ . From this point of view, the topology on  $\mathcal{M}$  can be thought of the “geometric limit topology” induced by the topologies on  $\tilde{\mathcal{M}}$  and  $\tilde{\mathcal{E}}$ . That is, a sequence of translation surfaces  $S_n$  with basepoints  $o_n$  should converge to  $S$  if the universal covers converge and the sequence of sets  $p_n^{-1}(o_n) \subset \tilde{S}_n$  converge to  $p_S^{-1}(o_S)$  within  $\tilde{\mathcal{E}}$ . This point of view is formalized by Theorem 17 below.

In the remainder of the section, we formally state results related to the ideas introduced above.

**Theorem 15.** *The map  $\mathcal{M} \rightarrow \tilde{\mathcal{M}}$  which sends a translation surface  $S$  to its universal cover  $\tilde{S}$  is continuous.*

For a translation surface  $S$ , let  $p_S : \tilde{S} \rightarrow S$  be the covering map. The domain of this map can most broadly be considered to be

$$(2) \quad \mathcal{P} = \{(S, \tilde{s}) \in \mathcal{M} \times \tilde{\mathcal{E}} : \tilde{s} \in \tilde{S}\}.$$

We say the *covering projection* is the map

$$p : \mathcal{P} \rightarrow \mathcal{E}; \quad (S, \tilde{s}) \mapsto p_S(\tilde{s}).$$

**Theorem 16** (Projection Theorem). *The covering projection is continuous.*

Finally, we state results describing convergence criteria and consequences of convergence in both  $\mathcal{M}$  and  $\mathcal{E}$ .

**Theorem 17** (Convergence in  $\mathcal{M}$ ). *Let  $\langle S_n \in \mathcal{M} \rangle$  be a sequence of translation surfaces with basepoints  $o_n \in S_n$ . Let  $p_n : \tilde{S}_n \rightarrow S_n$  be the universal covering maps. Then, the sequence  $\langle S_n \rangle$  converges if and only if the sequence of universal covers  $\tilde{S}_n \in \tilde{\mathcal{M}}$  converge to some  $\tilde{S} \in \tilde{\mathcal{M}}$  and there is a discrete set of points  $\tilde{O} \subset \tilde{S}$  such that the following statements hold:*

- (1) *For every  $\tilde{o} \in \tilde{O}$ , there is sequence  $\langle \tilde{o}_n \in p_n^{-1}(o_n) \rangle$  converging to  $\tilde{o}$  in  $\tilde{\mathcal{E}}$ .*
- (2) *For every increasing sequence of integers  $\langle n_k \rangle$  and every sequence of points  $\langle \tilde{o}_{n_k} \in p_{n_k}^{-1}(o_{n_k}) \rangle$  which converges to some point  $\tilde{o} \in \tilde{\mathcal{E}}$ , we have  $\tilde{o} \in \tilde{O}$ .*

*Moreover, if these statements hold, then  $\tilde{S}$  is the universal cover of  $S = \lim_{n \rightarrow \infty} S_n$ , and  $\tilde{O} = p_S^{-1}(o_S)$ .*

**Theorem 18** (Convergence in  $\mathcal{E}$ ). *Let  $\langle (S_n, s_n) \in \mathcal{E} \rangle$  be a sequence, and let  $(S, s) \in \mathcal{E}$ . Then,  $\langle (S_n, s_n) \rangle$  converges to  $(S, s)$  if the following statements are satisfied:*

- (1) *The sequence  $\langle S_n \in \mathcal{M} \rangle$  converges to  $S \in \mathcal{M}$ .*
- (2) *There is a sequence  $\langle \tilde{s}_n \in p_n^{-1}(s_n) \subset \tilde{\mathcal{E}} \rangle$  which converges to a point in  $p^{-1}(s) \subset \tilde{\mathcal{E}}$ .*

*Conversely, if  $\langle (S_n, s_n) \rangle$  converges to  $(S, s)$ , then  $S_n$  converges to  $S$  and for each  $\tilde{s} \in p_S^{-1}(s)$ , there is a sequence  $\langle \tilde{s}_n \in p_n^{-1}(s_n) \rangle$  which converges to  $\tilde{s}$ .*



## 6. OUTLINE OF REMAINDER OF PAPER

In §7, we investigate how a translation structure on a surface gives rise to a translation structure on its universal cover, and explore how this idea interacts with immersions.

In §8, we prove that the immersive topologies on  $\mathcal{M}$  and  $\mathcal{E}$  are second countable. This proves part of Theorem 8.

In §9, we investigate embeddings. We prove Theorems 5 and 6 as well as Proposition 7.

In §10, we further investigate the connection between a surface and its universal cover. We prove all the results in §5, and prove results which will allow us to use work done in [Hoo13a] for the remainder of the article. We also prove that the immersive topologies on  $\mathcal{M}$  and  $\mathcal{E}$  are Hausdorff, completing the proof of Theorem 8.

In §11, we investigate the action of moving the basepoint. We prove Theorems 10 and 11.

In §12, we prove that the  $GL(2, \mathbb{R})$  actions on  $\mathcal{M}$  and  $\mathcal{E}$  are continuous.

In §13, we prove Theorem 14, which states that the set of surfaces into which an open disk embeds is compact.

## 7. THE UNIVERSAL COVER AND IMMERSIONS

**7.1. Developing map.** Let  $\Delta$  denote the open topological disk with basepoint  $\tilde{x}_0$ . Suppose that  $\tilde{\mathcal{A}}$  is a translation atlas on  $\Delta$ . Then by analytic continuation, there is a unique local homeomorphism  $dev : \Delta \rightarrow \mathbb{R}^2$  so that:

- $dev(\tilde{x}_0) = \mathbf{0}$ .
- For each chart  $(\tilde{U}, \tilde{\phi}) \in \tilde{\mathcal{A}}$ , the map  $\tilde{\phi}$  differs from  $dev|_{\tilde{U}}$  locally only by translation. That is, for each  $\tilde{x} \in \tilde{U}$ , there is an open neighborhood  $\tilde{V}$  of  $\tilde{x}$  inside  $\tilde{U}$  so that  $\tilde{\phi}|_{\tilde{V}}$  and  $dev|_{\tilde{V}}$  differ by translation.

We call the map  $dev$  the *developing map* of the translation structure.

Let  $\tilde{\mathcal{M}} = \mathcal{M}(\Delta, \tilde{x}_0)$  and  $\tilde{\mathcal{E}} = \mathcal{E}(\Delta, \tilde{x}_0)$ . Following [Hoo13a], we call the fibers of the restricted projection

$$\tilde{\pi} = \pi|_{\tilde{\mathcal{E}}} : \tilde{\mathcal{E}} \rightarrow \tilde{\mathcal{M}}$$

*planar surfaces*, and denote them by letters such as  $P$  and  $Q$ . An alternate definition is that a planar surface is a translation surface which is homeomorphic to a disk. Each planar surface has an associated developing map. The union of these maps gives the *bundle-wide developing map*  $Dev : \tilde{\mathcal{E}} \rightarrow \mathbb{R}^2$ . We denote the individual developing maps of planar surfaces by restriction:  $Dev|_P : P \rightarrow \mathbb{R}^2$ .

**7.2. The universal cover.** A translation structure  $(X, x_0, \mathcal{A})$  induces a translation structure on the universal cover  $(\tilde{X}, \tilde{x}_0)$  of  $(X, x_0)$ . Let  $p : \tilde{X} \rightarrow X$  be the covering projection which satisfies  $p(\tilde{x}_0) = x_0$ . Given the translation atlas  $\mathcal{A}$  on  $(X, x_0)$ , consider the atlas

$$(3) \quad \{(\tilde{U}, \phi \circ p) : \tilde{U} = \pi^{-1}(U) \text{ and } (U, \phi) \in \mathcal{A}\}.$$

It can be observed that this new atlas is also a translation atlas, and thus it can be extended to a unique maximal translation atlas, which we denote by  $\tilde{\mathcal{A}}$ .

If  $S \in \mathcal{M}$  is a translation surface (with basepoint  $o_S$ ), its universal cover also inherits a translation structure as above. This universal cover is therefore isomorphic to a unique planar surface which we denote by  $\tilde{S} \in \tilde{\mathcal{M}}$ . We denote the basepoint of  $\tilde{S}$  by  $\tilde{o}_S$ . The isomorphism can be used to produce a covering map  $p_S : \tilde{S} \rightarrow S$  which satisfies  $p_S(\tilde{o}_S) = o_S$ .

**Remark 19.** We can recover a translation structure on a surface  $(X, x_0)$  from the developing map  $dev : \tilde{X} \rightarrow \mathbb{R}^2$ . Indeed, an open set  $\tilde{U} \subset \tilde{X}$  so that  $p_S|_{\tilde{U}} : \tilde{U} \rightarrow X$  is a homeomorphism onto its image, the pair  $(p_S(\tilde{U}), dev \circ (p_S|_{\tilde{U}})^{-1})$  is a compatible chart. The maximal collection of such charts recovers the translation structure. We view the developing map as easier to work than the translation atlas, but any statement which can be made in terms of the developing map can also be made in terms of the translation atlas.

In this paper, we will not be interested in immersing all types of subsets of translation surfaces. We will primarily be interested in

$$SC(S) = \{A \in PC(S) : A \text{ is locally path-connected and simply connected}\}.$$

We explain in some propositions why immersions are more natural with respect to sets in  $SC(S)$ .

**Proposition 20.** If  $A \in SC(S)$ , then there is an embedding  $\ell_A : A \hookrightarrow \tilde{S}$ . Moreover, the composition  $p_S \circ \ell_A$  is the identity on  $A$ .

We call the map  $\ell_A$  the *lifting map* and call  $\tilde{A} = \ell_A(A)$  the *lift* of  $A$ .

*Proof.* Since  $A$  is connected and locally path-connected, it admits a universal cover  $\tilde{A}$ . By general covering space theory, the inclusion of  $A$  into  $S$  lifts to a map  $\tilde{A} \rightarrow \tilde{S}$  so that  $\tilde{o}_S \mapsto o_S$ . Since  $A$  is simply connected, we can identify  $\tilde{A}$  with  $A$ . This gives our map  $\ell_A : A \rightarrow \tilde{S}$ . It is an immersion because  $p_S \circ \ell_A$  is the inclusion of  $A$  into  $S$ . In particular, it respects the translation structure defined by the charts given in equation 3.  $\square$

**Proposition 21.** Let  $S$  and  $T$  be a translation surfaces. Let  $A \in SC(S)$  and let  $B \subset T$  be an arbitrary subset containing the basepoint. The following statements are equivalent:

- (1) There is an immersion  $\iota : A \rightsquigarrow B$ .
- (2) There is an immersion  $\tilde{\iota} : \tilde{A} \rightarrow \tilde{T}$  with  $p_T \circ \tilde{\iota}(\tilde{A}) \subset B$ .
- (3) There is a continuous map  $\tilde{\iota} : \tilde{A} \rightarrow \tilde{T}$  with  $p_T \circ \tilde{\iota}(\tilde{A}) \subset B$  so that  $\tilde{\iota}(\tilde{o}_S) = \tilde{o}_T$  and  $Dev|_{\tilde{S}}(\tilde{s}) = Dev|_{\tilde{T}} \circ \tilde{\iota}(\tilde{s})$  for all  $\tilde{s} \in \tilde{A}$ .

Moreover, the maps  $\tilde{\iota}$  in statements (2) and (3) are the same.

The situation resulting from the existence of an immersion  $\iota : A \rightsquigarrow B$  with  $A \in SC(S)$  is summarized by the following commutative diagram:

$$\begin{array}{ccccc}
 & & \tilde{A} & \xrightarrow{p_S} & A \\
 & \swarrow Dev|_{\tilde{S}} & \downarrow \tilde{\iota} & \searrow \ell_A & \downarrow \iota \\
 \mathbb{R}^2 & & & & \\
 & \swarrow Dev|_{\tilde{T}} & \tilde{T} & \xrightarrow{p_T} & T
 \end{array}$$

We call the map  $\tilde{\iota}$  the *lifted immersion*.

*Proof.* Statements (2) and (3) are equivalent, because the translation structures are completely determined by the single charts  $(\tilde{S}, Dev|_{\tilde{S}})$  and  $(\tilde{T}, Dev|_{\tilde{T}})$ . Statement (3) simply restates the definition of immersion but restricted to consider these charts. These statements imply (1) since  $p_T \circ \tilde{\iota} \circ \ell_A$  is an immersion of  $A$  into  $B$ . Finally, given  $\iota$ , we can construct  $\tilde{\iota}$  by following the proof of the prior proposition with  $\iota$  replacing the inclusion of  $A$  into  $S$ .  $\square$

## 8. SECOND COUNTABILITY

Our proof of second countability of the topologies on  $\mathcal{M}$  and  $\mathcal{E}$  essentially follows from work done in §9 of [Hoo13a]. We offer an explicit countable subbasis for the immersive topologies on  $\mathcal{M}$  and  $\mathcal{E}$ .

We recall some definitions from §9 of [Hoo13a]. An open (resp. closed) *rectangle*  $R$  in a planar surface  $P$  is a subset so that the developing map restricted to  $R$  is a homeomorphism onto an open (resp. closed) rectangle in  $\mathbb{R}^2$ . We call a rectangle *rational* if the vertices of the image rectangle lie in  $\mathbb{Q}^2$ . An open (resp. closed) *rational rectangular union* in  $P$  is a finite union of open (resp. closed) rational rectangles in  $P$  which is connected and whose boundary consists of a union of closed curves.

**Theorem 22** (Second countability of  $\mathcal{M}$ ). *The collection of subsets of  $\mathcal{M}$  of the following four types give a countable subbasis for the immersive topology:*

- Sets of the form  $\mathcal{M}_\infty(D)$ , where  $D \in \overline{\text{Disk}}$  is a rational rectangular union.
- Sets of the form  $\mathcal{M}_{\neq}(U)$ , where  $U \in \text{Disk}$  is a rational rectangular union.
- Sets of the form  $\mathcal{M}_+(D, U)$ , where  $D \in \overline{\text{Disk}}$  is a rational rectangular union and  $U \subset D^\circ$  is an open rational rectangle.
- Sets of the form  $\mathcal{M}_-(D, K)$ , where  $D \in \overline{\text{Disk}}$  is a rational rectangular union and  $K \subset D$  is a finite union of closed rational rectangles.

*Proof.* The collection of all isomorphism classes of rational rectangular unions in planar surfaces is countable by Corollary 39 of [Hoo13a]. It follows that the subbasis described in the theorem is countable. It remains to prove that the subbasis above generates the immersive topology on  $\mathcal{M}$ . For the proof, we will call the topology generated by the sets listed in the theorem the *subbasis topology*. Clearly every open set in the subbasis topology is open in the immersive topology. We will prove that every open set used to define the immersive topology on  $\mathcal{M}$  is open in the subbasis topology.

First consider  $\mathcal{M}_\infty(D_1)$ , where  $D_1$  is an arbitrary closed disk. Suppose that  $S \in \mathcal{M}_\infty(D_1)$  so that there is an immersion  $\iota : D_1 \rightsquigarrow S$ . By Proposition 21, this is equivalent to the existence of an immersion  $\tilde{\iota} : D_1 \rightsquigarrow \tilde{S}$ . We can then apply Theorem 44 of [Hoo13a] to produce a rectangular union  $D_2 \in \overline{\text{Disk}}(\tilde{S})$  so that  $\tilde{\iota}(D_1) \subset D_2^\circ$ . By transitivity of immersions,  $D_1$  immerses in a surface whenever  $D_2$  does. Also,  $S \in \mathcal{M}_\infty(D_2)$  since the covering map  $p_S : \tilde{S} \rightarrow S$  restricts to an immersion  $p_S|_{D_2} : D_2 \rightsquigarrow S$ . Thus,  $S \in \mathcal{M}_\infty(D_2) \subset \mathcal{M}_\infty(D_1)$ , proving that the later is open in the subbasis topology.

Now suppose  $S \in \mathcal{M}_+(D_1, U_1)$ , where  $D_1$  is an arbitrary closed disk and  $U_1 \subset D_1$  is an arbitrary open subset. We repeat the argument above to construct  $D_2$ . Observe that  $\iota = p_S \circ \tilde{\iota}$ . Thus, there is a point  $\tilde{o} \in \tilde{\iota}(U_1)$  so that  $p_S(\tilde{o})$  is the basepoint  $o_S$  of  $S$ . Since  $\iota(U_1)$  is open, we can find an open rational rectangle  $U_2$  satisfying  $\tilde{o} \in U_2 \subset \tilde{\iota}(U_1)$ . Thus,  $S \in \mathcal{M}_+(D_2, U_2) \subset \mathcal{M}_+(D_1, U_1)$ , proving the later is open.

Now suppose that  $S \in \mathcal{M}_-(D_1, K_1)$ , where  $K_1 \subset D_1$  is an arbitrary closed set. Construct  $D_2$  as above, and note that  $o_S \notin p_S \circ \tilde{\iota}(K_1)$ . Equivalently, we have  $\tilde{\iota}(K_1) \cap p_S^{-1}(o_S) = \emptyset$ . The basepoint  $o_S$  has an  $\epsilon$  neighborhood which is isometric to a Euclidean disk. Then, no image of a metric ball of radius less than  $\frac{\epsilon}{2}$  in  $\tilde{S}$  under  $p_S$  can contain more than one point in  $p_S^{-1}(o_S)$ . Using compactness, we conclude that  $p_S^{-1}(o_S) \cap D_2$  is finite. Since  $\tilde{\iota}(K_1) \subset D_2^\circ$ , given any point  $\tilde{s} \in \tilde{\iota}(K_1)$ , we can find a closed rational rectangle  $R$  which does not intersect  $p_S^{-1}(o_S)$  and whose interior contains  $\tilde{s}$ . The interiors of these rectangles describe an open cover of the compact set  $\tilde{\iota}(K_1)$ , so there is a finite collection  $\mathcal{R}$  of such rectangles which cover  $\tilde{\iota}(K_1)$ .

Let  $K_2$  be the union of these rectangles, which contains  $\tilde{U}(K_1)$  and do not intersect  $p_S^{-1}(o_S)$ . Then,  $S \in \mathcal{M}_-(D_2, K_2) \subset \mathcal{M}_-(D_1, K_1)$ , which proves that  $\mathcal{M}_-(D_1, K_1)$  is open.

Finally, suppose that  $S \in \mathcal{M}_{\not\rightarrow}(U_1)$ , where  $U_1$  is an open disk. Then  $U_1 \not\rightarrow S$ , or equivalently  $U_1 \not\rightarrow \tilde{S}$ . By using a closed disk family for  $U_1$ , we can find a  $U_2 \subset U_1$  with compact closure so that  $U_2 \rightarrow S$ . (See Propositions 22 and 25 of [Hoo13a].) Then, we can find an open rational rectangular union  $U_3 \subset U_1$  which is homeomorphic to a disk so that  $U_2 \subset U_3$  using Theorem 44 of [Hoo13a]. We have  $U_3 \rightarrow S$ , since such an immersion would restrict to an immersion of  $U_2$  into  $S$ . Similarly, whenever  $U_1$  immerses into a surface, so does  $U_3$ . Thus,  $S \in \mathcal{M}_{\not\rightarrow}(U_3) \subset \mathcal{M}_{\not\rightarrow}(U_1)$ . This proves that  $\mathcal{M}_{\not\rightarrow}(U_1)$  is open.  $\square$

**Theorem 23** (Second countability of  $\mathcal{E}$ ). *A countable subbase for the immersive topology on  $\mathcal{E}$  is given by preimages under  $\pi : \mathcal{E} \rightarrow \mathcal{M}$  of a countable subbase for the topology on  $\mathcal{M}$  together with sets of the form  $\mathcal{E}_+(D, U)$ , where  $D \in \overline{\text{Disk}}$  is a rational rectangular union and  $U \subset D^\circ$  is an open rational rectangle.*

We omit the proof, because it very similar to the prior result. In particular, the proof that  $\mathcal{E}_+(D_1, U_1)$  is open in the topology generated by this subbasis is identical to the argument that  $\mathcal{M}_+(D_1, U_1)$  was open in the prior proof.

## 9. EMBEDDINGS

In this section, we prove the Embedding Theorem (Theorem 5), which stated that sets of the form  $\mathcal{M}_{\rightarrow}(D)$  are open whenever  $D$  is a closed disk in a translation surface. We also show that Proposition 7 (that  $\mathcal{E}_-(D, K)$  is open in  $\mathcal{E}$ ) follows.

**9.1. Ball embeddings.** Let  $B_\epsilon \subset \mathbb{R}^2$ , be the open ball of radius  $\epsilon$  centered at the origin, with the origin considered as the basepoint. Given a point  $s$  in a translation surface  $S$ , we define the *embedding radius* to be

$$ER(S, s) = \max\{\epsilon > 0 : B_\epsilon \hookrightarrow S^s\}.$$

Here,  $S^s$  denotes the translation surface which is translation isomorphic to  $S$ , but has the basepoint in the location of  $s \in S$ . See §4. This maximum is well defined unless  $S$  is the plane, in which case we take  $ER(S, s) = \infty$ .

Suppose  $\epsilon < ER(S, s)$ . Then, there is an embedding  $e : B_\epsilon \hookrightarrow S^s$ . Consider the basepoint changing isomorphism  $\beta_s : S \rightarrow S^s$ , which is a translation isomorphism and sends  $s \in S$  to the basepoint of  $S^s$ . We define the *ball embedding*

$$BE_s : B_\epsilon \rightarrow S; \quad \mathbf{v} \mapsto \beta_s^{-1} \circ e(\mathbf{v}).$$

As in [Hoo13a, §7.4], we also define some related quantities. If  $U \subset S$  is open, then it is translation isomorphic to a translation surface  $\hat{U}$ , and there is a natural embedding  $e : \hat{U} \rightarrow U$ . For  $s \in U$ , we define

$$ER(s \in U) = ER(\hat{U}, e^{-1}(s)).$$

If  $K \subset U$  is compact, then we define

$$ER(K \subset U) = \min_{s \in K} ER(s \in U).$$

This minimum is well defined, because the function  $s \mapsto ER(s \in U)$  is 1-Lipschitz.

**9.2. Proof of the Embedding Theorem.** We now prove the Embedding Theorem. We will briefly describe the main idea of the proof. Pick a closed disk  $D$ . The basic plan of the proof is to show that the Embedding Theorem is satisfied locally. By this we mean for every pair of points  $u$  and  $v$  in  $D$ , there are respective neighborhoods  $U$  and  $V$  and an open subset  $\mathcal{U} \subset \mathcal{M}$  so that if  $S \in \mathcal{I}$ , then there is an immersion  $\iota : D \rightsquigarrow S$  and this immersion looks like an embedding if we restrict attention to both  $U$  and  $V$ . This statements is made rigorous by the two lemmas below. The proof concludes by making an appeal to compactness of  $D \times D$ .

**Proposition 24.** *Let  $D$  be a closed disk in a planar surface  $P$ . Let  $\epsilon > 0$  be such that there is an embedding  $e : \bar{B}_\epsilon \rightsquigarrow D$ . Then, the following set is open in  $\mathcal{M}$ :*

$$\{S \in \mathcal{M} : \exists \iota : D \rightsquigarrow S \text{ and } (\iota \circ e)^{-1}(o_S) \cap \bar{B}_\epsilon = \{o_P\}\}.$$

*Proof.* Choose  $\epsilon \leq ER(o_P \in D^\circ)$ . Then, there is an embedding  $e : \bar{B}_\epsilon \hookrightarrow D$ . Let  $\mathcal{X} \subset \mathcal{M}$  be the set defined at the end of the proposition. Define

$$\mathcal{Y} = \mathcal{M}_-(D, \bar{B}_\epsilon \setminus B_{\frac{\epsilon}{2}}).$$

We claim that  $\mathcal{X} = \mathcal{Y}$ . This will prove the theorem, since  $\mathcal{Y}$  is a subbasis element of the topology on  $\mathcal{M}$ .

Observe that  $\mathcal{X} \subset \mathcal{Y}$ , since  $S \in \mathcal{X}$  implies that there is an immersion  $\iota : D \rightsquigarrow S$  and whenever  $p \in P$  is in  $e(\bar{B}_\epsilon) \setminus \{o_P\}$ ,  $\iota(p) \neq o_S$ . This in particular holds for points  $p \in \bar{B}_\epsilon \setminus B_{\frac{\epsilon}{2}}$ .

Now suppose that  $S \in \mathcal{Y}$  and  $S \notin \mathcal{X}$ . Then there is an immersion  $\iota : D \rightsquigarrow S$ , and there is a  $\mathbf{v} \in \bar{B}_\epsilon \setminus \{\mathbf{0}\}$  so that  $\iota \circ e(\mathbf{v}) = o_S$ . Since  $S \in \mathcal{Y}$ , it must be that  $\mathbf{v} \in B_{\frac{\epsilon}{2}} \setminus \{\mathbf{0}\}$ . The collection  $\{\iota \circ e(t\mathbf{v}) : 0 \leq t \leq 1\}$  is a closed geodesic on  $S$ . It follows that so long as  $n \in \mathbb{Z}$  and  $|n\mathbf{v}| \leq \epsilon$ , we have  $\iota \circ e(n\mathbf{v}) = o_S$ . Choose  $n$  to be the integer satisfying

$$\frac{\epsilon}{2} \leq n|\mathbf{v}| < \frac{\epsilon}{2} + |\mathbf{v}| < \epsilon.$$

Then,  $n\mathbf{v} \in \bar{B}_\epsilon \setminus B_{\frac{\epsilon}{2}}$  and  $\iota \circ e(n\mathbf{v}) = o_S$ . By definition  $S \notin \mathcal{Y}$ , which is a contradiction.  $\square$

**Lemma 25.** *Let  $P$  be a planar surface, let  $D_1, D_3 \in \overline{Disk}(P)$  with  $D_1 \subset D_3^\circ$ . Let  $u \in D_1$ . Then, there is a closed neighborhood  $\bar{U} \subset D_3^\circ$  of  $u$  so that the following set is open:*

$$\{S \in \mathcal{M} : \exists \iota : D_3 \rightsquigarrow S \text{ and } \iota|_{\bar{U}} \text{ is injective}\}.$$

*Proof.* Define the constant

$$\epsilon = \frac{1}{2} \min \{ER(D_1 \subset D_3^\circ), ER(o_P \in D_1)\}.$$

Then, there is an embedding  $e : \bar{B}_{2\epsilon} \hookrightarrow D_3$  be the associated embedding. Let  $\bar{U} \subset D_3$  be the closed  $\epsilon$ -ball centered at  $u \in D_1$ . Let  $\mathcal{X} \subset \mathcal{M}$  be the set defined at the end of the lemma. We must show that  $\mathcal{X}$  is open. In fact, we claim that  $\mathcal{X} = \mathcal{Y}$ , where

$$\mathcal{Y} = \{S \in \mathcal{M} : \exists \iota : D_3 \rightsquigarrow S \text{ and } (\iota \circ e)^{-1}(o_S) \cap \bar{B}_{2\epsilon} = \{o_P\}\}.$$

This set is open by Proposition 24.

We will prove that  $\mathcal{X} \subset \mathcal{Y}$ . Suppose that  $S \in \mathcal{M}$  and there is an immersion  $\iota : D_3 \rightsquigarrow S$ , but that  $S \notin \mathcal{Y}$ . We will show that  $S \notin \mathcal{X}$ . Since  $S \notin \mathcal{Y}$ , there is an  $\mathbf{v} \in B_{2\epsilon}$  with  $\mathbf{v} \neq \mathbf{0}$  so that  $\iota \circ e(\mathbf{v}) = o_S$ . Consider the map

$$(4) \quad f : D_1 \rightarrow D_3; \quad p \mapsto BE_p(\mathbf{v}).$$

Observe that the map  $\iota \circ f : D_1 \rightsquigarrow S$  is an immersion since  $f(o_P) = e(\mathbf{v})$ . So by uniqueness of immersions  $\iota \circ f = \iota$  on  $D_1$ . We claim that  $\iota|_{\bar{U}}$  is not injective. To see this observe that

$$\iota(BE_u(\frac{-\mathbf{v}}{2})) = \iota \circ f(BE_u(\frac{-\mathbf{v}}{2})) = \iota(BE_u(\frac{\mathbf{v}}{2})).$$

So, the observation that both  $BE_u(\frac{-\mathbf{v}}{2})$  and  $BE_u(\frac{\mathbf{v}}{2})$  are both within  $\bar{U}$  proves that  $\iota|_{\bar{U}}$  is not injective.

We now will show that  $\mathcal{Y} \subset \mathcal{X}$  via a similar argument. Let  $S$  be a translation surface so that there is an immersion  $\iota : D_3 \rightsquigarrow S$ . Suppose there are  $p, q \in \bar{U}$  so that  $\iota(p) = \iota(q)$ . Then,  $S \notin \mathcal{X}$  and we will show  $S \notin \mathcal{Y}$ . Let  $s \in S$  be this common image,  $s = \iota(p)$ . Consider the vector  $\mathbf{v} = e^{-1}(q) - e^{-1}(p)$ , which satisfies  $\mathbf{v} \neq 0$  and  $|\mathbf{v}| < 2\epsilon$ . Define  $f$  as in equation 4. We claim that  $\iota \circ f$  is an immersion of  $D_1$  into  $S$ . Since it is a local translation, it suffices to show that  $\iota \circ f(o_P) = o_S$ . Consider the basepoint changing isomorphism  $\beta_p : P \rightarrow P^p$  and  $\beta_s : S \rightarrow S^s$ . Let  $D_1^p = \beta_p(D_1)$  and  $D_3^p = \beta_p(D_3)$ . Then, the composition  $\beta_s \circ \iota \circ \beta_p^{-1}$  is an immersion of  $D_3^p$  into  $S^s$ . Consider precomposing with  $\beta_p \circ f \circ \beta_p^{-1} : D_1^p \rightarrow D_3^p$ . The composition is given by  $\beta_s \circ \iota \circ f \circ \beta_p^{-1}$  and is an immersion of  $D_1^p$  into  $S^s$ . By uniqueness of immersions, these maps agree on  $D_1^p$ . Thus, we have  $\iota \circ f = \iota$  on  $D_1$ . We conclude that

$$o_S = \iota(o_P) = \iota \circ f(o_P).$$

Since  $f(o_P) \in e(B_{2\epsilon})$ , we have that  $S \notin \mathcal{X}$ . □

**Lemma 26.** *Let  $P$  be a planar surface, let  $D_1, D_3 \in \overline{\text{Disk}}(P)$  with  $D_1 \subset D_3^\circ$ . Let  $u$  and  $v$  be distinct points in  $D_1$ . Let  $\bar{U}_\epsilon$  and  $\bar{V}_\epsilon$  be the closed  $\epsilon$  balls about  $u$  and  $v$ , respectively. Then, for sufficiently small  $\epsilon$ , the following set is open in  $\mathcal{M}$ :*

$$\{S \in \mathcal{M} : \exists \iota : D_3 \rightsquigarrow S \text{ and } \iota(\bar{U}) \cap \iota(\bar{V}) = \emptyset\}.$$

In order to prove the theorem, we will make use of the Fusion Theorem [Hoo13a, Theorem 18]. We state a variant of this result combining this theorem with Proposition 31 of [Hoo13a].

**Theorem 27** (Fusion Theorem). *Let  $P$  and  $Q$  be planar surfaces. Then, there is a unique planar surface  $R = P \curlywedge Q$  which satisfies the following statements:*

- $P \rightsquigarrow R$  and  $Q \rightsquigarrow R$ .
- For all trivial surfaces  $S$ , if  $P \rightsquigarrow S$  and  $Q \rightsquigarrow S$ , then  $R \rightsquigarrow S$ .

*Proof of Lemma 26.* Choose a  $D_2 \in \overline{\text{Disk}}(P)$  so that  $D_1 \subset D_2^\circ$  and  $D_2 \subset D_3^\circ$ . Choose

$$\epsilon < \frac{1}{2} \min \{ER(o_P \in D_1^\circ), ER(D_1 \subset D_2^\circ), ER(D_2 \subset D_3^\circ), d(u, v)\}.$$

Let  $\bar{U} \subset D_2^\circ$  be the closed ball of radius  $\epsilon$  about  $u$ , and let  $\bar{V} \subset D_2^\circ$  be the closed ball of radius  $\epsilon$  about  $v$ . Because  $2\epsilon < d(u, v)$ , these balls are disjoint.

We will now construct a new planar surface,  $Q$ . Let  $\beta_u : P \rightarrow P^u$  and  $\beta_v : P \rightarrow P^v$  be basepoint changing isomorphisms. Consider the open disks with alternate basepoints  $\beta_u(D_3^\circ)$  and  $\beta_v(D_2^\circ)$ . These surfaces are isomorphic to planar surfaces, and we define  $Q$  to be their fusion,

$$Q = \beta_u(D_3^\circ) \curlywedge \beta_v(D_2^\circ).$$

Associated to the fusion, we have immersions

$$\iota_u : \beta_u(D_2^\circ) \rightsquigarrow Q \quad \text{and} \quad \iota_v : \beta_v(D_3^\circ) \rightsquigarrow Q.$$



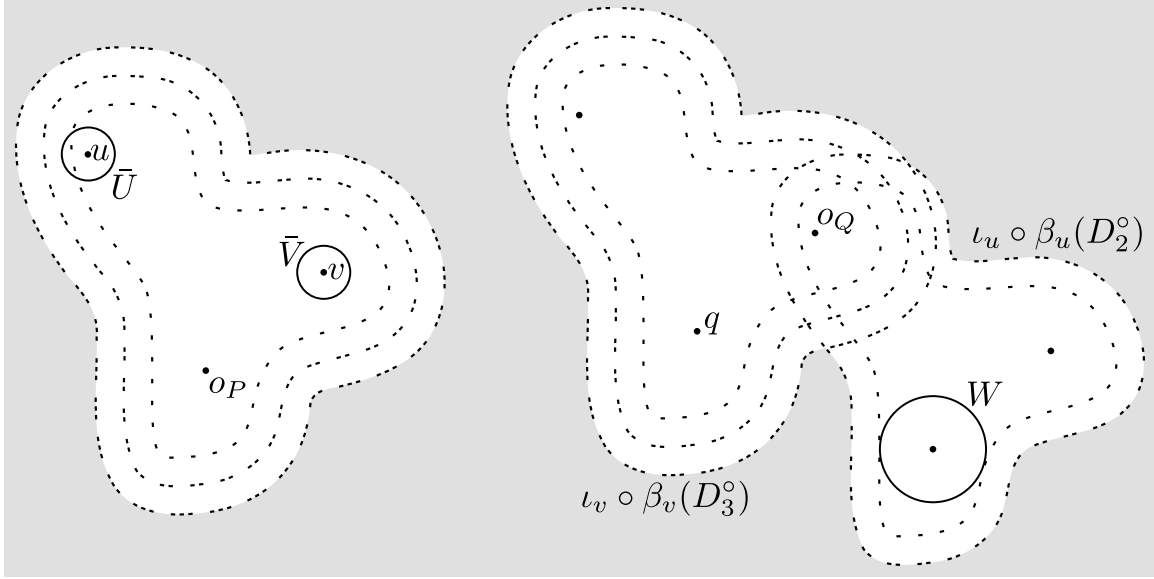


FIGURE 1. The left side of the figure shows the three disk  $D_1 \subset D_2 \subset D_3$  in  $P$  and related objects. The right side shows the fusion  $Q = \beta_u(D_3^\circ) \curlywedge \beta_v(D_2^\circ)$ .

Figure 1 depicts an example of  $Q$ , as well as the following objects. We define  $q = \iota_v \circ \beta_v(o_P)$ . We define  $\bar{W}$  to be the closed ball of radius  $2\epsilon$  about  $\iota_u \circ \beta_u(o_P)$ . We also choose a closed disk  $K \in \overline{\text{Disk}}(Q)$  which contains both  $\iota_u \circ \beta_u(D_1)$  and  $\iota_v \circ \beta_v(D_2)$ . Note that  $\bar{W} \subset \iota_u \circ \beta_u(D_1) \subset K$ .

We set  $\mathcal{X}$  to be the set defined at the end of the lemma. We will show that  $\mathcal{X}$  is open by proving that it is the same as the set

$$\mathcal{Y} = \mathcal{M}_{\rightsquigarrow}(D_3) \cap \left( \mathcal{M}_{\not\rightsquigarrow}(Q^q) \cup \mathcal{M}_-(K^q, W^q) \right).$$

Here,  $K^q = \beta_q(K)$  and  $W^q = \beta_q(W)$  represent subsets of  $Q^q$ , the surface  $Q$  with basepoint moved to the position of  $q$ . Let  $\mathcal{X}^c = \mathcal{M}_{\rightsquigarrow}(D_3) \setminus \mathcal{X}$  and  $\mathcal{Y}^c = \mathcal{M}_{\rightsquigarrow}(D_3) \setminus \mathcal{Y}$ . Since  $\mathcal{X}, \mathcal{Y} \subset \mathcal{M}_{\rightsquigarrow}(D_3)$ , is enough to prove that  $\mathcal{X}^c = \mathcal{Y}^c$ . Observe that these sets are defined by:

$$\mathcal{X}^c = \{S \in \mathcal{M} : \exists \iota : D_3 \rightsquigarrow S \text{ and } \iota(\bar{U}) \cap \iota(\bar{V}) \neq \emptyset\},$$

$$\mathcal{Y}^c = \{S \in \mathcal{M} : \exists \iota : D_3 \rightsquigarrow S, \exists j : Q^q \rightsquigarrow S, \text{ and } o_S \in j(W^q)\}.$$

(The equality for  $\mathcal{Y}^c$  uses the fact that  $W^q \subset K^q \subset Q^q$ . In particular, the immersion  $j : Q^q \rightsquigarrow S$  restricts to the immersion of  $K^q$  into  $S$ .)

We will begin by showing that  $\mathcal{X}^c \subset \mathcal{Y}^c$ . Let  $S \in \mathcal{X}^c$ . Then there is an immersion  $\iota : D_3 \rightsquigarrow S$  and there are points  $u' \in \bar{U}$  and  $v' \in \bar{V}$  so that  $\iota(u') = \iota(v')$ . The balls  $\bar{U}$  and  $\bar{V}$  are canonically identified with a closed ball of radius  $\epsilon$  in the plane. Using this identification, define the vectors  $\mathbf{u} = u' - u$  and  $\mathbf{v} = v' - v$ . Since  $u, v \in D_1$ , and  $2\epsilon < ER(D_1 \subset D_2^\circ)$ , there are ball embeddings

$$BE_u : \bar{B}_{2\epsilon} \rightarrow D_2^\circ \quad \text{and} \quad BE_v : \bar{B}_{2\epsilon} \rightarrow D_2^\circ.$$

By construction,  $u' = BE_u(\mathbf{u})$  and  $v' = BE_v(\mathbf{v})$ . Since, these points have the same image under  $\iota$ , so do the points  $u'' = BE_u(\mathbf{u} - \mathbf{v})$  and  $v$ . Let  $s = \iota(u'') = \iota(v)$ . We get two immersions into  $S^s$ . Since  $D_3 \rightsquigarrow S$ , we have  $\beta_v(D_3) = D_3^v \rightsquigarrow S^s$ . In addition, since  $\iota(u'') = s$ , we have an immersion  $k : \beta_{u''}(D_3) \rightsquigarrow S^s$ . Consider the following function:

$$f : D_2 \rightarrow D_3; \quad p \mapsto BE_p(\mathbf{u} - \mathbf{v}).$$

The map  $f$  respects the local translation structure and satisfies  $f(u) = u''$ . It follows that the following composition is an immersion:

$$k \circ \beta_{u''} \circ f \circ \beta_u^{-1} : \beta_u(D_2) \rightsquigarrow S^s.$$

In particular, we are using the structure the following commutative diagram:

$$(5) \quad \begin{array}{ccccc} D_2 & \xrightarrow{f} & D_3 & \rightsquigarrow^t & S \\ \downarrow \beta_u & & \downarrow \beta_{u''} & & \downarrow \beta_s \\ \beta_u(D_2) & \rightsquigarrow & \beta_{u''}(D_3) & \rightsquigarrow^k & S^s \end{array}$$

By restricting to the interiors, we have immersions  $\beta_u(D_2^\circ) \rightsquigarrow S^s$  and  $\beta_v(D_3^\circ) \rightsquigarrow S^s$ . It follows from the Fusion Theorem that we can immerse the fusion  $Q = \beta_u(D_2^\circ) \curlyvee \beta_v(D_3^\circ)$  into  $S^s$ .

We will now investigate properties of the immersion  $j' : Q \rightsquigarrow S^s$ . Recall that we set  $q = \iota_v \circ \beta_v(o_P) \in Q$ . We claim that  $j'(q) = \beta_s(o_S)$ . This follows from the commutative diagram:

$$\begin{array}{ccc} D_3 & \rightsquigarrow^t & S \\ \downarrow \beta_v & & \downarrow \beta_s \\ D_3^v & \rightsquigarrow^{\iota_v} & Q \rightsquigarrow^{j'} S^s \end{array}$$

It then follows that there is an immersion  $j : Q^q \rightsquigarrow S$ , and further immersions which make the following diagram commute:

$$\begin{array}{ccc} D_3 & \rightsquigarrow^t & S \\ \downarrow \beta_v & \searrow & \downarrow \beta_s \\ & Q^q & \\ \uparrow \beta_q & & \\ D_3^v & \rightsquigarrow^{\iota_v} & Q \rightsquigarrow^{j'} S^s \end{array}$$

We make a similar argument to the above to show that  $o_S \in j(W)$ . Observe that the immersion  $\beta_u(D_2^\circ) \rightsquigarrow S^s$  factors through  $Q$ . Modifying diagram 5, we have the following commutative diagram:

$$\begin{array}{ccc} D_2^\circ & \xrightarrow{f} & D_3 \rightsquigarrow^t S \\ \downarrow \beta_u & & \downarrow \beta_s \\ \beta_u(D_2^\circ) & \rightsquigarrow^{\iota_u} & Q \rightsquigarrow^{j'} S^s \\ & & \uparrow \beta_q \\ & & Q^q \end{array}$$

Note that the ball of radius  $2\epsilon$  centered at  $o_P$  sits inside of  $D_1^\circ$  and maps onto  $W$  under the map  $\iota_u \circ \beta_u$ . Consider the point  $r = BE_{o_P}(\mathbf{v} - \mathbf{u}) \in D_1$ . Since  $r$  is within  $2\epsilon$  of  $o_P$ , we know that  $\iota_u \circ \beta_u(r) \in W$ . This in turn is equivalent to the statement that  $\beta_q \circ \iota_u \circ \beta_u(r) \in W^q$ . It



remains to show that  $j \circ \beta_q \circ \iota_u \circ \beta_u(r) = o_S$ . This follows from commutativity of the diagram above since  $f(r) = o_P$ .

We will now show that  $\mathcal{Y}^c \subset \mathcal{X}^c$ . Choose  $S \in \mathcal{Y}^c$ . Then there is an immersion  $\iota : D_3 \rightsquigarrow S$  and an immersion  $j : Q^q \rightsquigarrow S$  so that  $o_S \in j(W^q)$ .

Because we defined  $s = \iota(v)$ , there is an immersion  $\iota' : \beta_v(D_3) \rightsquigarrow S^s$ . Since  $\iota_v : \beta_v(D_3^\circ) \rightsquigarrow Q$ , and  $\iota_v$  sends  $\beta_v(o_P)$  to  $q$ , we get an immersion  $\iota'_v : D_3^\circ \rightsquigarrow Q^q$ . Moreover, this immersion satisfies  $\iota'_v = \beta_q \circ \iota_v \circ \beta_v$ . In particular,  $\iota'_v(v) = \beta_q(o_Q)$ . By uniqueness of immersions, we get  $\iota|_{D_3^\circ} = j \circ \iota'_v$ . The map  $\beta_s \circ j \circ \beta_q$  sends  $o_Q$  to the basepoint  $\beta_s(s)$  of  $S^s$ , so this map is an immersion, which we call  $j'$ . The situation is summarized by following commutative diagram:

$$\begin{array}{ccccc} D_3^\circ & \xrightarrow{\iota'_v} & Q^q & \xrightarrow{j} & S \\ \downarrow \beta_v & & \beta_q \uparrow & & \downarrow \beta_s \\ \beta_v(D_3^\circ) & \xrightarrow{\iota_v} & Q & \xrightarrow{j'} & S^s \end{array}$$

By definition of  $Q$ , there is also an immersion  $\iota_u : \beta_u(D_2^\circ) \rightsquigarrow Q$ . We define  $\iota'_u = \beta_q \circ \iota_u \circ \beta_u$ . This is not an immersion, but respects the translation structure. We have the following commutative diagram:

$$\begin{array}{ccccc} D_2^\circ & \xrightarrow{\iota'_u} & Q^q & \xrightarrow{j} & S \\ \downarrow \beta_u & & \beta_q \uparrow & & \downarrow \beta_s \\ \beta_u(D_2^\circ) & \xrightarrow{\iota_u} & Q & \xrightarrow{j'} & S^s \end{array}$$

Since  $o_S \in j(W^q)$ , there is a  $w^q \in W^q$  so that  $j(w^q) = o_S$ . Let  $w = \beta_q^{-1}(w^q) \in W$ . By definition of  $W$ , there is a point  $p \in P$  so that  $d(p, o_P) \leq 2\epsilon$  and  $\iota_u \circ \beta_u(p) = w$ . Because  $2\epsilon < ER(o_P \in D_1^\circ)$ , we have  $p \in D_1^\circ$  and  $p = BE_{o_P}(2\mathbf{w})$  for some vector  $\mathbf{w} \neq 0$  with  $|\mathbf{w}| \leq \epsilon$ .

Let  $p_u = BE_{o_P}(\mathbf{w}) = BE_p(-\mathbf{w}) \in D_1$  and let  $p_v = BE_{o_P}(-\mathbf{w})$ . Since  $j \circ \iota'_u(p) = o_S$  and  $\iota(o_P) = o_S$ , we must also have  $j \circ \iota'_u(p_u) = \iota(p_v)$ . Consider the maps

$$f_u : D_1 \rightarrow D_2^\circ; * \mapsto BE_*(\mathbf{w}) \quad \text{and} \quad f_v : D_1 \rightarrow D_2^\circ; * \mapsto BE_*(-\mathbf{w}).$$

We claim that  $j \circ \iota'_u \circ f_u = \iota \circ f_v$ . This is because these maps respect the local translation structure, and they agree at one point:

$$j \circ \iota'_u \circ f_u(o_P) = j \circ \iota'_u(p_u) = \iota(p_v) = \iota \circ f_v(o_P).$$

In particular,  $j \circ \iota'_u \circ f_u(u) = \iota \circ f_v(u)$ . Observe that  $f_v(u) \in \bar{U}$ . We will use this equation to find a point in  $\bar{V}$  with the same image. Observe that  $\iota'_u(u) = \iota'_v(v) = \beta_q(o_Q)$ . Because these maps respect the translation structure, we can write:

$$\begin{aligned} j \circ \iota'_u \circ f_u(u) &= j \circ \iota'_u \circ BE_u(\mathbf{w}) = j \circ BE_{\iota'_u(u)}(\mathbf{w}) = j \circ BE_{\iota'_v(v)}(\mathbf{w}) \\ &= j \circ \iota'_v \circ BE_v(\mathbf{w}) = \iota \circ BE_v(\mathbf{w}). \end{aligned}$$

Here, we must be careful that whenever we commute  $BE(\mathbf{w})$  (say with  $\iota'_u$  for the second equal sign), we must check the corresponding point ( $u$  in this case) satisfies that the embedding radius into the domain of the subsequently applied map ( $j$  in this case) is larger than  $\epsilon$ . So for the second equals sign, this involves noting that  $u \in D_1$ ,  $\iota'_u$  is defined on  $D_2^\circ$  and  $ER(D_1 \subset D_2^\circ) > \epsilon$ . The conclusion is the observation that  $f_v(u) \in \bar{U}$ ,  $BE_v(\mathbf{w}) \in \bar{V}$ , and  $\iota \circ f_v(u) = \iota \circ BE_v(\mathbf{w})$ . This proves that  $\iota(\bar{U}) \cap \iota(\bar{V}) \neq \emptyset$ , and therefore  $S \in \mathcal{X}^c$ .  $\square$

We now prove the embedding theorem:

*Proof of Theorem 5.* Let  $D_1$  be a closed disk in a planar surface  $P$ . We will show that  $\mathcal{M}_{\hookrightarrow}(D)$  is open. Let  $T \in \mathcal{M}_{\hookrightarrow}(D_1)$  be a translation surface. Then  $D_1 \hookrightarrow T$  and we can find a  $D_3 \in \overline{\text{Disk}}(P)$  so that  $D_1 \subset D_3^\circ$  and  $D_3 \hookrightarrow T$ .

For each pair  $u, v \in D_1$ , we will use the lemmas above to produce a neighborhood  $U(u, v)$  of  $u$  and a neighborhood  $V(u, v)$  of  $v$  and an open set  $\mathcal{O}(u, v) \subset \mathcal{M}$ . If  $u \neq v$ , we construct  $U(u, v)$  and  $V(u, v)$  as in Lemma 26, and let  $\mathcal{O}(u, v) \subset \mathcal{M}$  be the open set produced. If  $u = v$ , we take  $U(u, v) = V(u, v)$  to be the set  $U$  in Lemma 25, and still define  $\mathcal{O}(u, v) \subset \mathcal{M}$  to be the open set produced.

The collection  $\{U^\circ(u, v) \times V^\circ(u, v) : u, v \in D_1\}$  forms an open covering of the compact set  $D_1 \times D_1$ . So, we can find a finite set of pairs  $(u_k, v_k)$  for  $k = 1, \dots, K$  so that

$$D_1 \times D_1 \subset \bigcup_{k=1}^K U_k \times V_k,$$

where  $U_k = U(u_k, v_k)$  and  $V_k = V(u_k, v_k)$ . We define the open set

$$\mathcal{U} = \bigcap_{k=1}^K \mathcal{O}_k, \quad \text{with } \mathcal{O}_k = \mathcal{O}(u_k, v_k).$$

Two claims about  $\mathcal{U}$  will prove the theorem. First, we claim that  $T \in \mathcal{U}$ . This is because  $D_3 \hookrightarrow T$ . It follows then from the statements of Lemma 26 and Lemma 25 that  $T \in \mathcal{O}(u, v)$  for all  $u, v \in D_1$ .

Second, we claim that if  $S \in \mathcal{U}$ , then  $D_1 \hookrightarrow S$ . Note that the statement  $S \in \mathcal{O}(u, v)$  for any  $u, v \in D_1$  implies that there is an immersion  $\iota : D_3 \rightsquigarrow S$ . We claim that the restriction  $\iota|_{D_1}$  is an embedding. Suppose otherwise. Then there are distinct  $u, v \in D_1$  so that  $\iota(u) = \iota(v)$ . Because of our covering, there is a  $k$  so that  $u \in U_k$  and  $v \in V_k$ . But the fact that  $T \in \mathcal{O}_k$  precludes the possibility of  $\iota(u) = \iota(v)$  by definition of  $\mathcal{O}_k$ .  $\square$

We now prove the disjointness theorem:

*Proof of Theorem 6.* The proof is similar in spirit to the prior proof. Let  $D$  be a closed disk, and let  $K_1, K_2 \subset D$  be closed and disjoint. We can assume without loss of generality that  $K_1$  and  $K_2$  lie in the interior of  $D$ . (Whenever a closed disk immerses in a surface, there is a slightly larger closed disk which also immerses.)

Let  $S \in \mathcal{M}_\emptyset(D; K_1, K_2)$ . Then, there is an immersion  $\iota : D \rightsquigarrow S$  and  $\iota(K_1) \cap \iota(K_2) = \emptyset$ . For each pair of points  $(u, v) \in K_1 \times K_2$ , we use Lemma 26 to choose closed balls  $\bar{U} = \bar{U}(u, v)$  about  $u$  and  $\bar{V} = \bar{V}(u, v)$  about  $v$  so that  $\iota(\bar{U}) \cap \iota(\bar{V}) = \emptyset$  and so that the following set is open:

$$\mathcal{O}(u, v) = \{S \in \mathcal{M} : \exists \iota : D \rightsquigarrow S \text{ and } \iota(\bar{U}) \cap \iota(\bar{V}) = \emptyset\}.$$

Then, the collection of interiors of sets in  $\{\bar{U}(u, v) \times \bar{V}(u, v) : (u, v) \in K_1 \times K_2\}$  covers  $K_1 \times K_2$ . By compactness, there is a finite collection of pairs  $\{(u_1, v_1), \dots, (u_k, v_k)\} \subset K_1 \times K_2$  so that

$$K_1 \times K_2 \subset \bigcup_{i=1}^k \bar{U}(u_i, v_i) \times \bar{V}(u_i, v_i).$$

We observe that

$$S \in \bigcap_{i=1}^k \mathcal{O}(u_i, v_i) \subset \mathcal{M}_\emptyset(D; K_1, K_2).$$

□

*Proof of Proposition 7.* Let  $D$  be a closed disk in a translation surface, and let  $K \subset D$  be a closed subset. We will show that  $\mathcal{E}_-(D, K)$  is open in  $\mathcal{E}$ .

Choose  $(S, s) \in \mathcal{E}_-(D, K)$ . Then, there is an immersion  $\iota : D \rightsquigarrow S$  and  $s \notin \iota(K)$ . We can therefore choose a closed metric ball  $B$  about  $s$  which is isometric to a Euclidean ball and disjoint from  $\iota(K)$ . By lifting, we also get an immersion  $\tilde{\iota} : D \rightsquigarrow \tilde{S}$ . Let  $p_S : \tilde{S} \rightarrow S$  be the universal covering map. Choose a  $\tilde{s} \in \tilde{S}$  so that  $p_S(\tilde{s}) = s$  and take  $\tilde{B} \subset \tilde{S}$  to be the closed ball of the same radius about  $\tilde{s}$ . Choose a  $D' \in \overline{\text{Disk}}(\tilde{S})$  to be large enough so that  $\tilde{B} \subset D'$  and  $\tilde{\iota}(D) \subset D'$ . We claim that

$$(S, s) \in \mathcal{E}_+(D_2, \tilde{B}^\circ) \cap \pi^{-1}(\mathcal{M}_\emptyset(D_2; \tilde{\iota}(K), \tilde{B})) \subset \mathcal{E}_-(D, K).$$

This will prove that  $\mathcal{E}_-(D, K)$  is open.

First, we prove that  $(S, s)$  lies in this intersection. The restriction of the covering map  $p_S$  gives an immersion  $D_2 \rightsquigarrow S$ . Moreover  $s \in B^\circ = p_S(\tilde{B}^\circ)$ . This proves that  $(S, s) \in \mathcal{E}_+(D_2, \tilde{B}^\circ)$ . In addition, by uniqueness of immersions we have  $\iota = p_S \circ \tilde{\iota}$ . So,  $p_S(\tilde{\iota}(K)) = \iota(K)$ , and we constructed  $B$  to be disjoint from  $\iota(K)$ . Thus,  $S \in \mathcal{M}_\emptyset(D_2; \tilde{\iota}(K), \tilde{B})$ .

Second, we must show that we have the inclusion. Suppose  $(T, t) \in \mathcal{E}_+(D_2, \tilde{B}^\circ)$  and  $T \in \mathcal{M}_\emptyset(D_2; \tilde{\iota}(K), \tilde{B})$ . Then, we get an immersion  $j : D_2 \rightsquigarrow T$ ,  $t \in j(\tilde{B}^\circ)$  and  $j \circ \tilde{\iota}(K) \cap j(\tilde{B}) = \emptyset$ . Since  $t \in j(\tilde{B})$ , we know that  $t \notin j \circ \tilde{\iota}(K)$ . Observe that  $j \circ \tilde{\iota}$  is the unique immersion of  $D$  into  $T$ , so this proves that  $(T, t) \in \mathcal{E}_-(D, K)$ . □

## 10. TRANSLATION STRUCTURES ON THE DISK

**10.1. Equivalence of topologies.** Recall that  $\tilde{\mathcal{M}} \subset \mathcal{M}$  is the collection of all isomorphism classes of translation structures on the disk, and that this space has been identified with the collection of all planar surfaces, which are fibers in  $\tilde{\mathcal{E}} \subset \mathcal{M}$ . The spaces  $\tilde{\mathcal{M}}$  and  $\tilde{\mathcal{E}}$  inherit subspace topologies from their inclusions into  $\mathcal{M}$  and  $\mathcal{E}$ , respectively. These spaces were also formally investigated in [Hoo13a], and were given a topology in that paper. The paper [Hoo13a] also placed topologies on  $\tilde{\mathcal{M}}$  and  $\tilde{\mathcal{E}}$ . In this subsection, we will prove that these topologies are the same.

**Lemma 28.** *The subspace topology on  $\tilde{\mathcal{M}}$  induced by the immersive topology on  $\mathcal{M}$  is the same as the topology on  $\tilde{\mathcal{M}}$  defined in [Hoo13a].*

*Proof.* Recall that a topology is a collection of sets (satisfying certain axioms), which by definition are open. For this proof, let  $\mathcal{I}$  denote the topology on  $\tilde{\mathcal{M}}$  as defined in this paper, and let  $\mathcal{I}'$  denote the topology on  $\tilde{\mathcal{M}}$  defined in [Hoo13a].

We will begin by explaining why  $\mathcal{I}' \subset \mathcal{I}$ . The topology on  $\mathcal{I}'$  was defined in [Hoo13a] to be the coarsest topology so that:

- Whenever  $K$  is a closed disk in a planar surface,  $\tilde{\mathcal{M}}_{\rightsquigarrow}(K) = \{P \in \tilde{\mathcal{M}} : K \rightsquigarrow P\}$  is open.
- Whenever  $U$  is an open disk in a planar surface,  $\tilde{\mathcal{M}}_{\not\rightsquigarrow}(U) = \{P \in \tilde{\mathcal{M}} : U \not\rightsquigarrow P\}$  is open.

Each set of the form  $\tilde{\mathcal{M}}_{\rightsquigarrow}(K)$  is open in  $\mathcal{I}$ , since  $\tilde{\mathcal{M}}_{\rightsquigarrow}(K) = \mathcal{M}_{\rightsquigarrow}(K) \cap \tilde{\mathcal{M}}$ .

Now let  $P \in \tilde{\mathcal{M}}_{\not\rightsquigarrow}(U)$ . We will show that there is an open set in  $\mathcal{I}$  which contains  $P$  and is contained in  $\tilde{\mathcal{M}}_{\not\rightsquigarrow}(U)$ . There are two possibilities. If  $U \not\rightsquigarrow P$ , then we have

$$P \in \tilde{\mathcal{M}} \cap \mathcal{M}_{\not\rightsquigarrow}(U) \subset \tilde{\mathcal{M}}_{\not\rightsquigarrow}(U),$$

and we conclude by noting that  $\tilde{\mathcal{M}} \cap \mathcal{M}_{\not\rightsquigarrow}(U) \in \mathcal{I}$ . Otherwise, there is an immersion  $\iota : U \rightsquigarrow P$ . In this case, we take an increasing family  $\{D_t \in \overline{\text{Disk}}(U)\}_{t>0}$  so that  $\bigcup_{t>0} D_t = U$ . Since  $U \not\rightsquigarrow P$ , there is a  $D \in \{D_t\}$  so that  $D \not\rightsquigarrow P$ . We claim that if  $\iota(D)$  immerses in a planar surface  $Q$ , then  $U \not\rightsquigarrow Q$ . Assume to the contrary that there is an immersion  $j : \iota(D) \rightsquigarrow Q$  and an embedding  $e : U \hookrightarrow Q$ . By the uniqueness of immersions, we have  $e|_D = j \circ \iota|_D$ . But,  $\iota|_D$  is not an embedding so  $e$  is also not. This proves that

$$(6) \quad P \in \tilde{\mathcal{M}}_{\rightsquigarrow}(\iota(D)) \subset \tilde{\mathcal{M}}_{\not\rightsquigarrow}(U).$$

But, it is not clear that  $\tilde{\mathcal{M}}_{\rightsquigarrow}(\iota(D)) \in \mathcal{I}$ , because  $\iota(D)$  need not be a disk. Let  $R \in \tilde{\mathcal{M}}_{\rightsquigarrow}(\iota(D))$ . Then there is an immersion  $k : \iota(D) \rightsquigarrow R$ . By continuity  $k \circ \iota(D)$  is compact, so we can find an  $E \in \overline{\text{Disk}}(R)$  so that  $k \circ \iota(D) \subset E$ . Then,

$$R \in \tilde{\mathcal{M}}_{\rightsquigarrow}(E) \subset \tilde{\mathcal{M}}_{\rightsquigarrow}(\iota(D)).$$

From the previous paragraph, we know  $\tilde{\mathcal{M}}_{\rightsquigarrow}(E) \in \mathcal{I}$ , which implies that  $\tilde{\mathcal{M}}_{\rightsquigarrow}(\iota(D)) \in \mathcal{I}$ . Then, it follows from equation 6 that  $\tilde{\mathcal{M}}_{\rightsquigarrow}(\iota(D)) \in \mathcal{I}$ .

Now we will show that  $\mathcal{I} \subset \mathcal{I}'$ . As in the prior proof, we work through our subbasis of  $\mathcal{I}$ . This consists of intersections with  $\tilde{\mathcal{M}}$  of the four types of sets described in §3.2 for the subbasis for topology on  $\mathcal{M}$ . We work through these four types one at a time. First, let  $D$  be a closed disk in a translation surface. Then  $D$  has a lift  $\tilde{D}$  to its universal cover by Proposition 20. From above, and because  $D$  and  $\tilde{D}$  are indistinguishable via immersions (Proposition 4), we have

$$\tilde{\mathcal{M}} \cap \mathcal{M}_{\rightsquigarrow}(D) = \tilde{\mathcal{M}} \cap \mathcal{M}_{\rightsquigarrow}(\tilde{D}) = \tilde{\mathcal{M}}_{\rightsquigarrow}(\tilde{D}).$$

Second, let  $U$  be an open disk in a translation surface. Again, let  $\tilde{U}$  be the lift to the surface's universal cover. We have

$$\tilde{\mathcal{M}} \cap \mathcal{M}_{\not\rightsquigarrow}(U) = \tilde{\mathcal{M}} \cap \mathcal{M}_{\not\rightsquigarrow}(\tilde{U}) = \{P \in \tilde{\mathcal{M}} : \tilde{U} \not\rightsquigarrow P\}.$$

This last set is denoted  $\tilde{\mathcal{M}}_{\not\rightsquigarrow}(\tilde{U})$  in [Hoo13a], and this set is open by [Hoo13a, Theorem 8].

To handle the remaining two basis elements of  $\mathcal{I}$ , we must recall two facts from [Hoo13a]. First, the canonical section of the projection  $\tilde{\pi} : \tilde{\mathcal{E}} \rightarrow \tilde{\mathcal{M}}$  is given by

$$\tilde{\sigma} : \tilde{\mathcal{M}} \rightarrow \tilde{\mathcal{E}}; \quad P \mapsto (P, o_P)$$

is continuous by [Hoo13a, Proposition 11]. Second, for any closed disk  $\tilde{D}$  in a planar surface, any  $\tilde{U} \subset \tilde{D}^\circ$  open, and any  $\tilde{K} \subset \tilde{D}$  closed, the following two sets are open:

$$(7) \quad \tilde{\mathcal{E}}_+(\tilde{D}, \tilde{U}) = \{(P, p) \in \tilde{\mathcal{E}} : \exists \iota : \tilde{D} \rightsquigarrow P \text{ and } p \in \iota(\tilde{U})\}.$$

$$(8) \quad \tilde{\mathcal{E}}_-(\tilde{D}, \tilde{K}) = \{(P, p) \in \tilde{\mathcal{E}} : \exists \iota : \tilde{D} \rightsquigarrow P \text{ and } p \notin \iota(\tilde{K})\}.$$

Sets of the form  $\tilde{\mathcal{E}}_+(\tilde{D}, \tilde{U})$  form the subbasis for the topology on  $\tilde{\mathcal{E}}$  defined in [Hoo13a, §4.3], and sets of the form  $\tilde{\mathcal{E}}_-(\tilde{D}, \tilde{K})$  are open by [Hoo13a, Proposition 12].

Now we consider the third and fourth types of open sets for the topology  $\mathcal{I}$ . Let  $D$  be a closed disk in a translation surface, let  $\tilde{U} \subset D^\circ$  be open, and let  $K \subset D$  be closed. Let  $\ell_D : D \hookrightarrow \tilde{D}$  be the lift of  $D$  to the universal cover of the translation surface containing  $D$  as in Proposition 20. Let  $\tilde{U} = \ell_D(U)$  and  $\tilde{K} = \ell_D(K)$ . Observe that we have the following identities:

$$\begin{aligned}\tilde{\mathcal{M}} \cap \mathcal{M}_+(D, U) &= \tilde{\mathcal{M}} \cap \mathcal{M}_+(\tilde{D}, \tilde{U}) = \tilde{\sigma}^{-1}(\tilde{\mathcal{E}}_+(\tilde{D}, \tilde{U})). \\ \tilde{\mathcal{M}} \cap \mathcal{M}_-(D, K) &= \tilde{\mathcal{M}} \cap \mathcal{M}_-(\tilde{D}, \tilde{K}) = \tilde{\sigma}^{-1}(\tilde{\mathcal{E}}_-(\tilde{D}, \tilde{K})).\end{aligned}$$

It follows that both sets are open in  $\mathcal{I}'$  by the continuity of  $\tilde{\sigma}$ .  $\square$

**Lemma 29.** *The subspace topology on  $\tilde{\mathcal{E}}$  induced by the immersive topology on  $\mathcal{E}$  is the same as the topology on  $\tilde{\mathcal{E}}$  defined in [Hoo13a].*

*Proof.* For this proof, let  $\mathcal{J}$  denote the topology on  $\tilde{\mathcal{E}}$  given as the subspace topology inherited from the immersive topology as defined in this paper, and let  $\mathcal{J}'$  denote the topology on  $\tilde{\mathcal{E}}$  defined in [Hoo13a].

We will begin by explaining why  $\mathcal{J}' \subset \mathcal{J}$ . The topology is defined in  $\mathcal{J}'$  is the coarsest topology so that  $\tilde{\pi} : \tilde{\mathcal{E}} \rightarrow \tilde{\mathcal{M}}$  is continuous, and so that sets of the form  $\tilde{\mathcal{E}}_+(\tilde{D}, \tilde{U})$ , defined in equation 7 are open. Recall that the projection  $\pi : \mathcal{E} \rightarrow \mathcal{M}$  is defined to be continuous in the topology on  $\mathcal{E}$ , and we have  $\tilde{\pi} = \pi|_{\tilde{\mathcal{E}}}$ . Thus,  $\tilde{\pi}$  is continuous in the topology  $\mathcal{J}$ . Also observe that for any closed disk  $\tilde{D}$  in a planar surface, and any  $\tilde{U} \subset \tilde{D}^\circ$  open, we have

$$\tilde{\mathcal{E}}_+(\tilde{D}, \tilde{U}) = \tilde{\mathcal{E}} \cap \mathcal{E}_+(\tilde{D}, \tilde{U}).$$

The set  $\mathcal{E}_+(\tilde{D}, \tilde{U})$  is a subbasis element of  $\mathcal{J}$ .

Now we will show that  $\mathcal{J} \subset \mathcal{J}'$ . By definition  $\mathcal{J}$  is the coarsest topology so that  $\pi|_{\tilde{\mathcal{E}}}$  is continuous, and for any closed disk  $D$  in a translation surface any  $U \subset D^\circ$  open, and  $K \subset D$  closed, we have that  $\tilde{\mathcal{E}} \cap \mathcal{E}_+(D, U)$ . Again  $\pi|_{\tilde{\mathcal{E}}} = \tilde{\pi}$ , which is defined to be continuous in  $\mathcal{J}'$ . Fix  $D, U$  and  $K$  as above. Define  $\tilde{D}, \tilde{U}$  and  $\tilde{K}$  as in the last paragraph of the previous proof. Then, we have:

$$\tilde{\mathcal{E}} \cap \mathcal{E}_+(D, U) = \tilde{\mathcal{E}} \cap \mathcal{E}_+(\tilde{D}, \tilde{U}) = \tilde{\mathcal{E}}_+(\tilde{D}, \tilde{U}).$$

Here, the set on the right hand side is as in equation 7, and are in  $\mathcal{J}'$  as discussed surrounding that equation.  $\square$

**10.2. Continuity and the universal cover.** Since the subspace topology defined on  $\tilde{\mathcal{M}} \subset \mathcal{M}$  is the same as the topology defined in [Hoo13a], we note that this topology has a particularly nice subbasis. We restate Corollary 9 of [Hoo13a]:

**Corollary 30** (Subbasis of  $\tilde{\mathcal{M}}$ ). *A subbasis for the topology on  $\tilde{\mathcal{M}} \subset \mathcal{M}$  is given by sets of the following two forms:*

- Sets of the form  $\tilde{\mathcal{M}}_{\rightsquigarrow}(K) = \{Q \in \tilde{\mathcal{M}} : K \rightsquigarrow Q\}$  where  $K \in \overline{Disk}(P)$  for some  $P \in \tilde{\mathcal{M}}$ .
- Sets of the form  $\tilde{\mathcal{M}}_{\not\rightsquigarrow}(U) = \{Q \in \tilde{\mathcal{M}} : U \not\rightsquigarrow Q\}$  where  $U \in Disk(P)$  for some  $P \in \tilde{\mathcal{M}}$ .

Let  $v : \mathcal{M} \rightarrow \tilde{\mathcal{M}}$  be the map which sends a translation surface  $S$  to its universal cover  $\tilde{S}$ . We show this map is continuous as claimed in Theorem 15.

*Proof of Theorem 15.* We will check that the preimage of the subbasis of open sets provided by Corollary 30 is open. By Proposition 21, the statement that a simply connected set immerses in a surface  $S$  is logically equivalent to the statement that the set immerses in  $\tilde{S}$ . Thus,  $v^{-1}(\tilde{\mathcal{M}}_{\rightsquigarrow}(K)) = \mathcal{M}_{\rightsquigarrow}(K)$  for all closed disks  $K$ , and  $v^{-1}(\tilde{\mathcal{M}}_{\neq}(U)) = \mathcal{M}_{\neq}(U)$  for all open disks  $U$ . These sets are open by definition of the immersive topology on  $\mathcal{M}$ .  $\square$

**10.3. The Hausdorff property.** We will now prove that the immersive topologies on  $\mathcal{M}$  and  $\mathcal{E}$  are Hausdorff:

*Proof of Theorem 8.* We proved that the topologies on  $\mathcal{M}$  and  $\mathcal{E}$  are second countable in §8. It remains to prove that the topologies are Hausdorff.

We begin by showing  $\mathcal{M}$  is a Hausdorff space. Let  $S$  and  $T$  be distinct points in  $\mathcal{M}$ . We will prove that they can be separated by disjoint open sets.

First, suppose that their universal covers  $\tilde{S}$  and  $\tilde{T}$  are distinct. By Theorem [Hoo13a, Theorem 14], the topology on  $\tilde{\mathcal{M}}$  is Hausdorff. So, there are disjoint open sets  $\mathcal{U}$  and  $\mathcal{V}$  in  $\tilde{\mathcal{M}}$  so that  $\tilde{S} \in \mathcal{U}$  and  $\tilde{T} \in \mathcal{V}$ . By Theorem 15, the sets  $v^{-1}(\mathcal{U})$  and  $v^{-1}(\mathcal{V})$  are open. They separate  $S$  and  $T$ .

Now suppose that  $\tilde{S} = \tilde{T}$ . Let  $p_S : \tilde{S} \rightarrow S$  and  $p_T : \tilde{S} \rightarrow T$  be the associated covering maps. By distinctness, we can assume without loss of generality that there is a  $\tilde{s} \in \tilde{S}$  so that  $p_S(\tilde{s}) = o_S$  but  $p_T(\tilde{s}) \neq o_T$ . Choose a  $D \in \overline{\text{Disk}}(\tilde{S})$  so that  $\tilde{s} \in D^\circ$ . By discreteness of the lifts of the basepoint of  $T$  to  $\tilde{S}$ , we can choose an open set  $U \subset D^\circ$  so that  $\tilde{s} \in U$  and  $\bar{U} \cap p_T^{-1}(o_T) = \emptyset$ . Then we have  $S \in \mathcal{M}_+(D, U)$  and  $T \in \mathcal{M}_-(D, \bar{U})$ . These sets are open by definition of the immersive topology on  $\mathcal{M}$ . Moreover, they are disjoint since  $R \in \mathcal{M}_-(D, \bar{U})$  implies that there is an immersion  $\iota : D \rightarrow R$  with  $o_R \notin \iota(\bar{U})$ . In this case,  $o_R \notin \iota(U)$  so  $R \notin \mathcal{M}_+(D, U)$ .

Now we will prove that the immersive topology on  $\mathcal{E}$  is Hausdorff. Let  $(S, s)$  and  $(T, t)$  be distinct points in  $\mathcal{E}$ . If  $S \neq T$ , then since  $\mathcal{M}$  is Hausdorff, we can find disjoint open sets  $\mathcal{U}, \mathcal{V} \subset \mathcal{M}$  so that  $S \in \mathcal{U}$  and  $T \in \mathcal{V}$ . Then,  $(S, s) \in \pi^{-1}(\mathcal{U})$  and  $(T, t) \in \pi^{-1}(\mathcal{V})$ . These sets are disjoint. The definition of the topology on  $\mathcal{E}$  guarantees that  $\pi$  is continuous, so these sets are also open.

Now suppose  $S = T$ . Then,  $s, t \in S$  are distinct. Choose lifts  $\tilde{s}, \tilde{t} \in \tilde{S}$ . Then we can find a closed disk  $D \in \overline{\text{Disk}}(\tilde{S})$  so that  $\tilde{s}, \tilde{t} \in D^\circ$ . Then, the open set  $p_S(D^\circ) \subset S$  contains both  $s$  and  $t$ . Since  $p_S(D^\circ)$  is an open set in a surface, it is Hausdorff, so we can find an open set  $U$  containing  $s$  so that  $t \notin \bar{U}$ . Then,  $(S, s) \in \mathcal{E}_+(D, U)$  and  $(S, t) \in \mathcal{E}_-(D, \bar{U})$ . These sets are disjoint by the same reasoning as we used to conclude that  $\mathcal{M}$  is Hausdorff.  $\square$

#### 10.4. Proof of the Projection Theorem.

*Proof of Theorem 16.* We will prove the covering projection,  $p : (S, \tilde{s}) \mapsto p_S(\tilde{s}) \in \mathcal{E}$  is continuous. Since  $\pi \circ p : (S, \tilde{s}) \mapsto S$  is continuous, it suffices to show that  $p^{-1}(\mathcal{E}_+(D, U))$  is open for every closed disk  $D$  and every open  $U \subset D^\circ$ .

Suppose that  $(S, \tilde{s}) \in p^{-1}(\mathcal{E}_+(D, U))$ . Let  $s = p(S, \tilde{s}) = p_S(\tilde{s})$ . Then, there is an immersion  $\iota : D \rightsquigarrow S$  and  $s \in \iota(U)$ . It follows that there is an immersion  $\tilde{\iota} : D \rightsquigarrow \tilde{S}$ , and there is a lift  $\tilde{s}_*$  of  $s$  inside of  $\tilde{\iota}(U)$ . Since  $p_S(\tilde{s}) = p_S(\tilde{s}_*)$ , there is an element of the deck group of the covering,  $\gamma : \tilde{S} \rightarrow \tilde{S}$ , so that  $\gamma(\tilde{s}_*) = \tilde{s}$ . Let  $\tilde{o}_* = \gamma(\tilde{o}_S)$  be the image of the basepoint of  $\tilde{S}$ .

Choose a  $\tilde{D} \in \overline{\text{Disk}}(\tilde{S})$  so that  $\tilde{s} \in \tilde{D}^\circ$ ,  $\tilde{o}_* \in \tilde{D}^\circ$ , and  $\gamma \circ \tilde{\iota}(D) \subset \tilde{D}^\circ$ . Choose

$$\epsilon < \min \left\{ ER(\tilde{s}_* \in \tilde{D}^\circ), \frac{1}{2} ER(\tilde{s}_* \in \tilde{\iota}(U)), ER(\gamma \circ \tilde{\iota}(D) \subset \tilde{D}^\circ) \right\}.$$

Let  $\tilde{B}_o$  and  $\tilde{B}_s$  denote the open balls in  $\tilde{S}$  of radius  $\epsilon$  about  $\tilde{o}_*$  and  $\tilde{s}$ , respectively. Observe that  $S \in \mathcal{M}_+(\tilde{D}, \tilde{B}_o)$  and  $\tilde{s} \in \mathcal{E}_+(\tilde{D}, \tilde{B}_s)$ .

We claim that if  $(T, \tilde{t})$  satisfies  $T \in \mathcal{M}_+(\tilde{D}, \tilde{B}_o)$  and  $\tilde{t} \in \tilde{T} \cap \mathcal{E}_+(\tilde{D}, \tilde{B}_s)$ , then  $t = p_T(\tilde{t}) \in \mathcal{E}_+(D, U)$ . Once this is proved, we see that  $\mathcal{M}_+(\tilde{D}, \tilde{B}_o) \times \mathcal{E}_+(\tilde{D}, \tilde{B}_s)$  intersected with the domain of the projection is an open subset of the domain which contains  $(S, \tilde{s})$  and is contained in  $p^{-1}(\mathcal{E}_+(D, U))$ .

Now we will prove our claim. Since  $\tilde{t} \in \tilde{T} \cap \mathcal{E}_+(\tilde{D}, \tilde{B}_s)$ , there is an immersion  $\tilde{j} : \tilde{D} \rightsquigarrow \tilde{T}$  and  $\tilde{t} \in \tilde{j}(\tilde{B}_s)$ . By postcomposing with the covering map  $p_T$ , we also get an immersion  $j = p_T \circ \tilde{j} : \tilde{D} \rightsquigarrow T$ , and  $o_T \in j(\tilde{B}_o)$  since  $T \in \mathcal{M}_+(\tilde{D}, \tilde{B}_o)$ . This means we can find a point  $\tilde{o}_+ \in \tilde{j}(\tilde{B}_o) \cap p_T^{-1}(o_T)$ . Then, there is a vector  $\mathbf{v}$  with norm less than  $\epsilon$  so that

$$\tilde{o}_+ = BE_{\tilde{j}(\tilde{o}_*)}(\mathbf{v}).$$

Define the function

$$f : \gamma \circ \tilde{i}(D) \rightarrow \tilde{D}; \quad x \mapsto BE_x(\mathbf{v}).$$

We claim that  $j \circ f \circ \gamma \circ \tilde{i} : D \rightarrow T$  is an immersion. To see this note that each map locally respects the translation structure, and the composition respects the basepoints,

$$j \circ f \circ \gamma \circ \tilde{i}(o_D) = p_T \circ \tilde{j} \circ f \circ \gamma \circ \tilde{i}(o_D) = p_T(\tilde{o}_+) = o_T.$$

Since  $\tilde{t} \in \mathcal{E}_+(\tilde{D}, \tilde{B}_s)$ , we know that  $\tilde{t}$  is within  $\epsilon$  of  $\tilde{j}(\tilde{s})$ . We also have  $\tilde{j}(\tilde{s}) = \tilde{j} \circ \gamma(\tilde{s}_*)$ . Therefore,  $\tilde{j}(\tilde{s})$  is within  $\epsilon$  of  $\tilde{j} \circ f \circ \gamma(\tilde{s}_*)$ . So, by the triangle inequality,  $\tilde{t}$  is within  $2\epsilon$  of  $\tilde{j} \circ f \circ \gamma(\tilde{s}_*)$ . Since  $2\epsilon < ER(\tilde{s}_* \in \tilde{i}(U))$ , we conclude that  $\tilde{t} \in \tilde{j} \circ f \circ \gamma \circ \iota(U)$ . Therefore,  $t \in j \circ f \circ \gamma \circ \iota(U)$ , which proves that  $t \in \mathcal{E}_+(D, U)$  as desired.  $\square$

The joint continuity of immersions is a consequence of Theorem 16 and work in [Hoo13a].

*Proof of Proposition 9.* Fix an open disk  $U$ . Let  $\mathcal{I}(U) \subset \mathcal{M}$  be the set of  $S \in \mathcal{M}$  so that there is an immersion  $\iota_S : U \rightsquigarrow S$ . Recall that  $I_U : \mathcal{I}(U) \times U \rightarrow \mathcal{E}$  is defined by  $I_U(S, u) = \iota_S(u)$ . Let  $\tilde{\mathcal{I}}(U) = \mathcal{I}(U) \cap \tilde{\mathcal{M}}$ , and let  $\tilde{I}_U : \tilde{\mathcal{I}}(U) \times U \rightarrow \tilde{\mathcal{E}}$  be the restriction of  $I_U$  to  $\tilde{\mathcal{I}}(U)$ . Proposition 13 of [Hoo13a] states that for any open disk  $U$ , the map  $\tilde{I}_U$  is continuous. The continuity of  $I_U$  then follows, because whenever  $S \in \mathcal{I}(U)$ , we have  $\tilde{S} \in \tilde{\mathcal{I}}(U)$  and  $I_U(S, u) = p_S \circ \tilde{I}_U(\tilde{S}, u)$ , where  $p_S : \tilde{S} \rightarrow S$  is the covering map. Here, we are using both the continuity of the map  $S \mapsto \tilde{S}$  and of the covering projection (Theorem 16).  $\square$

**10.5. Convergence results.** Before proving our convergence results, we provide a lemma which produces a quotient translation surface from the set of lifts of the basepoint of the surface to its universal cover.

**Lemma 31.** *Let  $\tilde{S} \in \tilde{\mathcal{M}}$  and suppose  $\tilde{O} \subset \tilde{S}$  satisfies the following statements:*

- (1) *The set  $\tilde{O}$  is discrete as a subset of  $\tilde{S}$ .*
- (2) *The collection  $\Gamma = \{\beta_{\tilde{o}} : \tilde{o} \in \tilde{O}\}$  of basepoint changing isomorphisms forms a group of translation automorphisms of  $\tilde{S}$ .*

*Then, there is a surface  $S \in \mathcal{M}$  with universal cover  $\tilde{S}$  so that  $p_S^{-1}(o_S) = \tilde{O}$ .*

In the proof, we will use the fact that the basepoint changing map and basepoint changing isomorphism are continuous when restricted to  $\tilde{\mathcal{M}}$  and  $\tilde{\mathcal{E}}$ . The following is a restatement of Theorem 16 of [Hoo13a].



**Theorem 32.** *The restriction of the basepoint changing map,  $BC|_{\tilde{\mathcal{E}}} : \tilde{\mathcal{E}} \rightarrow \tilde{\mathcal{M}}$  is continuous. The basepoint changing isomorphism  $q \mapsto \beta_p(q)$  is jointly continuous when restricted to*

$$\{(p, q) \in \tilde{\mathcal{E}}^2 : \pi(p) = \pi(q)\}.$$

*The inverse basepoint changing isomorphism,  $r \mapsto \beta_p^{-1}(r)$  is continuous when restricted to*

$$\{(p, r) \in \tilde{\mathcal{E}}^2 : \pi(r) = BC(p)\}.$$

*Proof of Lemma 31.* Our surface  $S$  will be the quotient  $\tilde{S}/\Gamma$ . First assume that this quotient is a translation surface  $S$ . Let  $\tilde{o}_S$  be the basepoint of  $\tilde{S}$ . By statement (2), the maps  $\beta_{\tilde{o}}$  are all translation automorphisms. Thus, they are of the form  $\beta_{\tilde{o}} : \tilde{S} \rightarrow \tilde{S}$ . Because the identity automorphism must be in  $\Gamma$ , we know that the base point of  $\tilde{S}$ ,  $\tilde{o}_S$ , lies in  $\tilde{O}$ . Also observe by definition of  $\beta$  that  $\beta_{\tilde{o}}(\tilde{o}) = \tilde{o}_S$ . This proves the last remark that  $p_S^{-1}(o_S) = \tilde{O}$ , assuming that the quotient is a translation surface.

To prove that the quotient is a translation surface, we will show that each point  $\tilde{s} \in \tilde{S}$  lies in an open set  $U \subset \tilde{S}$  so that whenever  $\beta_1, \beta_2 \in \Gamma$  satisfy  $\beta_1(U) \cap \beta_2(U) \neq \emptyset$ , we have  $\beta_1 = \beta_2$ . Assume that  $\tilde{s}$  does not have this property. Let  $B_\epsilon$  denote the open  $\epsilon$  ball about  $\tilde{s}$ . Then, for each  $\epsilon$ , there is a distinct pair  $\beta_1, \beta_2 \in \Gamma$  so that  $\beta_1(B_\epsilon) \cap \beta_2(B_\epsilon) \neq \emptyset$ . In other words,  $B_\epsilon \cap \beta_1^{-1} \circ \beta_2(B_\epsilon) \neq \emptyset$ . In particular, the distance from  $\tilde{s}$  to  $\beta_1^{-1} \circ \beta_2(\tilde{s})$  is less than  $2\epsilon$ . By statement (2),  $\beta_1^{-1} \circ \beta_2 \in \Gamma$ . So there is an  $\tilde{o} = \tilde{o}(\epsilon) \in \tilde{O}$  so that  $\beta_1^{-1} \circ \beta_2 = \beta_{\tilde{o}}$ . Applying this for the sequence  $\epsilon = \frac{1}{n}$  produces a sequence  $\tilde{o}_n \in \tilde{O}$  so that

$$\lim_{n \rightarrow \infty} \beta_{\tilde{o}_n}(\tilde{s}) = \tilde{s}.$$

Let  $\tilde{s}_n = \beta_{\tilde{o}_n}(\tilde{s})$ . Then,  $\beta_{\tilde{o}_n}$  is a translation automorphism of  $\tilde{S}$  which carries  $\tilde{s}$  to  $\tilde{s}_n$ . It follows that  $BC(\tilde{S}, \tilde{s}) = BC(\tilde{S}, \tilde{s}_n)$ . Let  $\tilde{T}$  denote this common surface, and consider the maps

$$\beta_{\tilde{s}} : \tilde{S} \rightarrow \tilde{T} \quad \text{and} \quad \beta_{\tilde{s}_n} : \tilde{S} \rightarrow \tilde{T}.$$

By definition, the first map carries  $\tilde{s}$  to the basepoint of  $\tilde{T}$ , and the second carries  $\tilde{s}_n$  to the basepoint of  $\tilde{T}$ . We claim that

$$\beta_{\tilde{o}_n} = \beta_{\tilde{s}_n}^{-1} \circ \beta_{\tilde{s}}.$$

By the above remarks, both send  $\tilde{s}$  to  $\tilde{s}_n$ . Since there can be only one translation automorphism which does this, they must be equal. We can recover the sequence  $\langle \tilde{o}_n \rangle$  as

$$\tilde{o}_n = \beta_{\tilde{o}_n}^{-1}(\tilde{o}_S) = \beta_{\tilde{s}}^{-1} \circ \beta_{\tilde{s}_n}(\tilde{o}_S).$$

By Theorem 32, we have

$$\lim_{n \rightarrow \infty} \tilde{o}_n = \lim_{n \rightarrow \infty} \beta_{\tilde{s}}^{-1} \circ \beta_{\tilde{s}_n}(\tilde{o}_S) = \beta_{\tilde{s}}^{-1} \circ \beta_{\tilde{s}}(\tilde{o}_S) = \tilde{o}_S.$$

Since  $\beta_{\tilde{o}_n}$  is never the identity translation automorphism, the existence of the sequence  $\langle \tilde{o}_n \in \tilde{O} \rangle$  approaching  $\tilde{o}_S$  violates the discreteness of  $\tilde{O}$ . (Note that  $\tilde{o}_S$  is necessarily in  $\tilde{O}$ , because the group  $\Gamma$  needs an identity element.)  $\square$

We now prove our convergence criterion for  $\mathcal{M}$ .

*Proof of Theorem 17.* Let  $\langle S_n \in \mathcal{M} \rangle$  be a sequence of translation surfaces converging to  $S$ . Let  $o \in S$  and  $o_n \in S_n$  denote basepoints, and let  $p : \tilde{S} \rightarrow S$  and  $p_n : \tilde{S}_n \rightarrow S_n$  denote the universal covering maps. We will prove that statements (1) to (2) of the theorem hold for  $\tilde{O} = p^{-1}(o)$ ,



Recall that the map  $v : \mathcal{M} \rightarrow \tilde{\mathcal{M}}$ , which sends a surface to its universal cover, is continuous by Theorem 15. Thus  $\langle \tilde{S}_n \rangle$  converges to  $\tilde{S}$ .

Choose  $\tilde{o} \in p^{-1}(o)$ . To prove (1), we must find a sequence  $\tilde{o}_n \in p_n^{-1}(o_n)$  so that  $\langle \tilde{o}_n \rangle$  converges to  $\tilde{o}$  in  $\tilde{\mathcal{E}}$ . Choose a closed disk  $D \in \overline{\text{Disk}}(\tilde{S})$  so that  $\tilde{o} \in D^\circ$ . Since  $\langle \tilde{S}_n \rangle$  converges to  $\tilde{S}$ , there is an  $N$  so that for  $n > N$ , there is an immersion  $\tilde{v}_n : D \rightsquigarrow \tilde{S}_n$ . Note that  $p_n \circ \tilde{v}_n : D \rightsquigarrow S$  is an immersion. Choose an  $\epsilon > 0$  small enough so that the open ball  $B_\epsilon$  of radius  $\epsilon$  about  $\tilde{o}$  is contained in  $D$ . Since  $\mathcal{M}_+(D, B_\epsilon)$  is open and contains  $S$ , there is an  $M > N$  so that  $o_n \in p_n \circ \tilde{v}_n(B_\epsilon)$  for  $n > M$ . It follows that for  $n > M$ , there is a point in  $B_\epsilon$  so that  $o_n$  is the image of the point under  $p_n \circ \tilde{v}_n$ . Now let  $\epsilon \rightarrow 0$ . We can choose a closest lift  $p_n \in D^\circ$  to  $\tilde{o}$  so that  $p_n \circ \tilde{v}_n(p_n) = o_n$  and so that the sequence  $\langle p_n \rangle$  converges to  $\tilde{o}$  inside of  $D^\circ$ . It then follows from joint continuity of immersions (Proposition 9) that  $\tilde{o}_n = \tilde{v}_n(p_n)$  converges to  $\tilde{o}$  inside of  $\tilde{\mathcal{E}}$ .

Let  $\langle n_k \rangle$  be an increasing sequence of positive integers. Let  $\langle \tilde{o}_{n_k} \in \tilde{S}_{n_k} \rangle$  be a sequence which converges to  $\tilde{o} \in \tilde{\mathcal{E}}$  and satisfies  $p_{n_k}(\tilde{o}_{n_k}) = o_{n_k}$ . To show (2), we must prove that  $p(\tilde{o})$  is the basepoint  $o \in S$ . Since  $\langle \tilde{S}_{n_k} \rangle$  converges to  $\tilde{S}$ , we know by continuity of  $\pi : \mathcal{E} \rightarrow \mathcal{M}$  that  $\tilde{o} \in \tilde{S}$ . We also know that  $\langle S_{n_k} \rangle$  converges to  $S$ . Therefore, by the Projection Theorem  $\langle o_{n_k} = p_{n_k}(\tilde{o}_{n_k}) \rangle$  tends to  $p(\tilde{o})$ . Since each  $o_{n_k}$  is the basepoint of  $S_{n_k}$ , the limit  $p(\tilde{o})$  must be the basepoint of  $S$ . (This statement about basepoints follows from Proposition 9, for instance.)

Now we will prove the converse. Suppose that  $\langle S_n \in \mathcal{M} \rangle$  is a sequence of surfaces. Let  $\langle \tilde{S}_n \in \tilde{\mathcal{M}} \rangle$  be the sequence of universal covers, and let  $\tilde{S} \in \tilde{\mathcal{M}}$  be another surface in  $\tilde{\mathcal{M}}$ . Suppose  $\tilde{O} \subset \tilde{\mathcal{M}}$  is a discrete subset satisfying the statements (1) and (2) of the theorem. We will check that this set  $\tilde{O}$  satisfies the two statements of Lemma 31. Observe that the first statement (that  $\tilde{O}$  is discrete), is tautologically satisfied.

We will now prove the second statement of Lemma 31. Pick a  $\tilde{q} \in \tilde{O}$ . We will first show that  $\beta_{\tilde{q}}$  is an automorphism of  $\tilde{S}$ . This requires showing that  $\text{BC}(\tilde{S}, \tilde{q}) = \tilde{S}$ . By statement (1) of the theorem, there is a sequence  $\langle \tilde{q}_n \in p_n^{-1}(o_n) \rangle$  which converges to  $\tilde{q}$ . Then by continuity of BC, observe that

$$\text{BC}(\tilde{S}, \tilde{q}) = \lim_{n \rightarrow \infty} \text{BC}(\tilde{S}_n, \tilde{q}_n) = \lim_{n \rightarrow \infty} \tilde{S}_n = \tilde{S}.$$

It remains to show that  $\Gamma = \{\beta_{\tilde{q}} : \tilde{q} \in \tilde{O}\}$  forms a group. First observe that  $\Gamma$  contains the identity element, because the basepoint  $\tilde{o}$  of  $\tilde{S}$  lies in  $\tilde{O}$ . Observe that the basepoint  $\tilde{o}_n$  of  $\tilde{S}_n$  lies in  $p_n^{-1}(o_n)$  for every  $n$ . Since  $\langle \tilde{S}_n \rangle$  tends to  $\tilde{S}$ ,  $\langle \tilde{o}_n \rangle$  tends to  $\tilde{o}$ . So by statement (2) of the Theorem,  $\tilde{o} \in \tilde{O}$ .

Since the identity element lies in  $\Gamma$ , it suffices to prove that for any pair of elements  $\gamma_1, \gamma_2 \in \Gamma$ , we also have  $(\gamma_2 \gamma_1)^{-1} \in \Gamma$ . We will use the fact that

$$\Gamma_n = \{\beta_{\tilde{q}_n} : \tilde{q}_n \in p_n^{-1}(o_n)\}$$

has this property since it is the deck group of the cover  $p_n : \tilde{S}_n \rightarrow S_n$ . Choose two elements  $\tilde{q}, \tilde{r} \in \tilde{O}$ . We will take  $\gamma_1 = \beta_{\tilde{q}}$  and  $\gamma_2 = \beta_{\tilde{r}}$ . By statement (1) of the theorem, there are sequences  $\langle \tilde{q}_n \in p_n^{-1}(o_n) \rangle$  and  $\langle \tilde{r}_n \in p_n^{-1}(o_n) \rangle$  which converge to  $\tilde{q}$  and  $\tilde{r}$ , respectively. By Theorem 32, we know that

$$(9) \quad \beta_{\tilde{r}} \circ \beta_{\tilde{q}}(\tilde{o}) = \lim_{n \rightarrow \infty} \beta_{\tilde{r}_n} \circ \beta_{\tilde{q}_n}(\tilde{o}_n),$$

where  $\tilde{o}$  and  $\tilde{o}_n$  denote the basepoints of  $\tilde{S}$  and  $\tilde{S}_n$ , respectively. Define

$$\tilde{s} = \beta_{\tilde{r}} \circ \beta_{\tilde{q}}(\tilde{o}) \quad \text{and} \quad \tilde{s}_n = \beta_{\tilde{r}_n} \circ \beta_{\tilde{q}_n}(\tilde{o}_n).$$

Observe that by definition of  $\beta$ , we have

$$\beta_{\tilde{s}}(\tilde{s}) = \tilde{o} \quad \text{and} \quad \beta_{\tilde{s}_n}(\tilde{s}_n) = \tilde{o}_n.$$

A translation automorphism is determined by the image of a single point. So, we can conclude that

$$\beta_{\tilde{s}} = (\beta_{\tilde{r}} \circ \beta_{\tilde{q}})^{-1} \quad \text{and} \quad \beta_{\tilde{s}_n} = (\beta_{\tilde{r}_n} \circ \beta_{\tilde{q}_n})^{-1}.$$

So, it suffices to prove that  $\tilde{s} \in \tilde{O}$ . Since  $\Gamma_n$  is a group,  $\beta_{\tilde{s}_n} \in \Gamma$ . It follows that  $\tilde{s}_n \in p_n^{-1}(o_n)$ . Equation 9 showed that  $\langle \tilde{s}_n \rangle$  converges to  $\tilde{s}$ , so by statement (2) of the theorem, we know that  $\tilde{s} \in \tilde{O}$ . This concludes the proof that  $\Gamma$  is a group.  $\square$

We now prove our criterion for convergence in  $\mathcal{E}$ .

*Proof of Theorem 18.* First suppose that  $\langle (S_n, s_n) \in \mathcal{E} \rangle$  converges to  $(S, s) \in \mathcal{E}$ . Since  $\pi : \mathcal{E} \rightarrow \mathcal{M}$  is continuous,  $\langle S_n \rangle$  converges to  $S$ . Now let  $p : \tilde{S} \rightarrow S$  be the universal covering map. Choose an arbitrary  $\tilde{s} \in p^{-1}(s)$  and choose a  $D \in \overline{\text{Disk}}(\tilde{S})$  so that  $\tilde{s} \in D^\circ$ . Let  $\epsilon > 0$  be small enough so that the open ball  $B_\epsilon \subset \tilde{S}$  of radius  $\epsilon$  about  $\tilde{s}$  lies in  $D^\circ$ . Observe that  $(S, s) \in \mathcal{E}_+(D, B_\epsilon)$ . Since this set is open, there is an  $N = N(\epsilon)$  so that  $(S_n, s_n) \in \mathcal{E}_+(D, B_\epsilon)$  for  $n > N$ . That is, there is an immersion  $\iota_n : D \rightsquigarrow S_n$  and  $s_n \in \iota_n(B_\epsilon)$ . In other words,  $B_\epsilon \cap \iota_n^{-1}(s_n) \neq \emptyset$  for  $n > N(\epsilon)$ . Let  $q_n$  be a choice of closest point in  $\iota_n^{-1}(s_n)$  to  $\tilde{s}$  for  $n > N(1)$ . By letting  $\epsilon$  tend to zero in the remarks above, we see that  $q_n$  converges to  $\tilde{s}$  within  $D^\circ$ . The immersions  $\iota_n$  lift to immersions  $\tilde{\iota}_n : D \rightsquigarrow \tilde{S}_n$ . Let  $\tilde{s}_n = \tilde{\iota}_n(q_n)$  for  $n > N(1)$ . We have  $\iota_n = p_n \circ \tilde{\iota}_n$ , where  $p_n : \tilde{S}_n \rightarrow S_n$  is the covering map. Thus,  $p_n(\tilde{s}_n) = s_n$ . By continuity of immersions applied to the immersions  $\iota_n : D \rightsquigarrow S_n$  converging to the restriction of the identity on  $D \subset S$ , we see that  $\tilde{s}_n = \tilde{\iota}_n(q_n)$  tends to  $\tilde{s}$ .

Now suppose statements (1) and (2) hold for the sequence  $\langle (S_n, s_n) \in \mathcal{E} \rangle$  and  $(S, s) \in \mathcal{E}$ . We will prove that the sequence converges to  $(S, s)$ . By statement (1), we know that  $\langle S_n \rangle$  converges to  $S$ . By statement (2), there is a sequence  $\langle \tilde{s}_n \in p_n^{-1}(s_n) \rangle$  converging to a point  $\tilde{s} \in p^{-1}(s)$ . So, by the Projection Theorem,  $s_n = p_n(\tilde{s}_n)$  converges to  $s = p(\tilde{s})$ .  $\square$

## 11. BASEPOINT CHANGE

Recall that if  $(S, s) \in \mathcal{E}$ , then  $\text{BC}(S, s) = S^s \in \mathcal{M}$  is the translation surface which is isomorphic to the surface  $S$  with the basepoint relocated to  $s \in S$ . We also have basepoint changing isomorphisms  $\beta_s : S \rightarrow S^s$ . Our goal here is to prove Theorems 10 and 11, which claim that these maps are continuous and jointly continuous, respectively.

We will utilize work done in [Hoo13a], which already proved these results for translation structures on disks. We now make a basic observation that describes how we move to the general case. Let  $(S, s) \in \mathcal{E}$ . Let  $p : \tilde{S} \rightarrow S$  be the universal covering map, and choose  $\tilde{s} \in p^{-1}(s)$ . Then, we can consider the basepoint changing isomorphisms  $\beta_s : S \rightarrow S^s$  and  $\tilde{\beta}_{\tilde{s}} : \tilde{S} \rightarrow \tilde{S}^{\tilde{s}}$ . We observe that  $\text{BC}(\tilde{S}, \tilde{s}) = \tilde{S}^{\tilde{s}}$  is the universal cover of  $\text{BC}(S, s) = S^s$ . Let

$p' : \tilde{S}^{\tilde{s}} \rightarrow S^s$  be the universal covering map. Then, we have the commutative diagram:

$$(10) \quad \begin{array}{ccc} \tilde{S} & \xrightarrow{\tilde{\beta}_{\tilde{s}}} & \tilde{S}^{\tilde{s}} \\ \downarrow p & & \downarrow p' \\ S & \xrightarrow{\beta_s} & S^s \end{array}$$

We now prove that the basepoint changing map,  $\text{BC} : (S, s) \mapsto S^s$ , is continuous.

*Proof of Theorem 10.* Let  $\langle (S_n, s_n) \in \mathcal{E} \rangle$  be a sequence converging to  $(S, s)$ . Let  $S'_n = \text{BC}(S_n, s_n) \in \mathcal{M}$  and  $S' = \text{BC}(S, s)$ . We need to show that  $\langle S'_n \rangle$  converges to  $S'$ .

Let  $p_n : \tilde{S}_n \rightarrow S_n$ ,  $p'_n : \tilde{S}'_n \rightarrow S'_n$ ,  $p : \tilde{S} \rightarrow S$ , and  $p' : \tilde{S}' \rightarrow S'$  be universal coverings. By Theorem 18, there is a sequence of lifts  $\langle \tilde{s}_n \in p_n^{-1}(s_n) \rangle$  converging to a  $\tilde{s} \in p^{-1}(s)$ . By Theorem 32,  $\tilde{S}'_n = \text{BC}(\tilde{S}_n, \tilde{s}_n) \in \tilde{\mathcal{M}}$  converges to  $\tilde{S}' = \text{BC}(\tilde{S}, \tilde{s}) \in \tilde{\mathcal{M}}$ . Let  $\tilde{\beta}_n : \tilde{S}_n \rightarrow \tilde{S}'_n$  and  $\tilde{\beta} : \tilde{S} \rightarrow \tilde{S}'$  be the basepoint changing isomorphisms which carry  $\tilde{s}_n$  and  $\tilde{s}$  to the basepoints of  $\tilde{S}'_n$  and  $\tilde{S}'$ , respectively. Theorem 32 also implies that if a sequence  $\langle \tilde{t}_n \in \tilde{S}_n \rangle$  tends to  $\tilde{t} \in \tilde{S}$ , then  $\langle \tilde{\beta}_n(\tilde{t}_n) \in \tilde{S}'_n \rangle$  tends to  $\tilde{\beta}(\tilde{t}) \in \tilde{S}'$ . In addition, if a sequence  $\langle \tilde{t}'_n \in \tilde{S}'_n \rangle$  tends to  $\tilde{t}' \in \tilde{S}'$ , then  $\langle \tilde{\beta}_n^{-1}(\tilde{t}'_n) \in \tilde{S}_n \rangle$  tends to  $\tilde{\beta}^{-1}(\tilde{t}') \in \tilde{S}$ .

To prove  $\langle S'_n \rangle$  converges to  $S'$ , we will utilize the criterion for convergence given by Theorem 17. We begin by verifying condition (1). Let  $\tilde{t}' \in \tilde{S}'$  be a lift of the basepoint of  $S'$ . Then,  $\tilde{t} = \tilde{\beta}^{-1}(\tilde{t}')$  satisfies  $p(\tilde{t}) = s$ , where  $p : \tilde{S} \rightarrow S$  is the universal covering map. By Theorem 18, there is a sequence  $\langle \tilde{t}_n \in p_n^{-1}(s_n) \rangle$  which converges to  $\tilde{t} \in \tilde{S}$ . By Theorem 32, the sequence  $\langle \tilde{t}'_n = \tilde{\beta}_n(\tilde{t}_n) \rangle$  converges to  $\tilde{t}' = \tilde{\beta}(\tilde{t})$ . By the commutative diagram in equation 10, each  $p'_n(\tilde{t}'_n)$  is the basepoint of  $S'_n$ . Thus, the fact that  $\langle \tilde{t}'_n \rangle$  tends to  $\tilde{t}'$  verifies condition (1).

Now we will check condition (2). Let  $\langle n_k \rangle$  be an increasing sequence of integers. Let  $\langle \tilde{t}'_{n_k} \in \tilde{S}'_{n_k} \rangle$  be a sequence of lifts of the basepoints of  $S'_{n_k}$ , and suppose that the sequence converges to  $\tilde{t}' \in \tilde{\mathcal{E}}$ . Since  $\langle \tilde{S}'_{n_k} \rangle$  converges to  $\tilde{S}'$ , we know that  $\tilde{t}' \in \tilde{S}'$ . Let  $\tilde{t}_{n_k} = \tilde{\beta}_{n_k}^{-1}(\tilde{t}'_{n_k})$  and  $\tilde{t} = \tilde{\beta}^{-1}(\tilde{t}')$ . By Theorem 32,  $\langle \tilde{t}_{n_k} \rangle$  tends to  $\tilde{t}$ . By the commutative diagram in equation 10, each  $p_{n_k}(\tilde{t}_{n_k}) = s_{n_k}$ . Therefore, the Projection Theorem implies  $p(\tilde{t}) = s$ . Then again by commutativity,  $\tilde{t}' = \tilde{\beta}(\tilde{t})$  is a lift of the basepoint of  $S'$ . This verifies condition (2).  $\square$

We will now prove the joint continuity of the basepoint changing isomorphism and its inverse.

*Proof of Theorem 11.* Let  $\langle (S_n, s_n) \in \mathcal{E} \rangle$  be a sequence converging to  $(S, s) \in \mathcal{E}$ . Let  $S'_n = \text{BC}(S_n, s_n)$  and  $S' = \text{BC}(S, s)$ . Let  $\beta_n : S_n \rightarrow S'_n$  and  $\beta : S \rightarrow S'$  be the associated basepoint changing isomorphisms.

To show that the basepoint changing isomorphism is jointly continuous, it suffices to prove that if  $\langle t_n \in S_n \rangle$  converges to  $t \in S$ , then  $\langle \beta_n(t_n) \rangle$  converges to  $\beta(t)$ .

We consider the universal covers using notation from the prior proof. By Theorem 18, there are sequences of lifts  $\langle \tilde{s}_n \in p_n^{-1}(s_n) \rangle$  converging to a  $\tilde{s} \in p^{-1}(s)$  and  $\langle \tilde{t}_n \in p_n^{-1}(s_n) \rangle$  converging to a  $\tilde{t} \in p^{-1}(s)$ . Let  $\tilde{S}'_n = \text{BC}(\tilde{S}_n, \tilde{s}_n)$  and  $\tilde{S}' = \text{BC}(\tilde{S}, \tilde{s})$ . Let  $\tilde{\beta}_n : \tilde{S}_n \rightarrow \tilde{S}'_n$  and  $\tilde{\beta} : \tilde{S} \rightarrow \tilde{S}'$  be the associated basepoint changing isomorphisms. By Theorem 32, we know that  $\langle \tilde{\beta}_n(\tilde{t}_n) \in \tilde{S}'_n \rangle$  converges to  $\tilde{\beta}(\tilde{t})$ . By the Theorem 10, we know that  $\langle \tilde{S}'_n \rangle$  tends to  $\tilde{S}'$  in  $\mathcal{M}$ . So we can apply the Projection Theorem to conclude that  $\langle p'_n \circ \tilde{\beta}_n(\tilde{t}_n) \in \tilde{S}'_n \rangle$  tends to  $p' \circ \tilde{\beta}(\tilde{t}) \in \tilde{S}'$ . Then by the commutative diagram in equation 10, we see this is the same as saying that  $\langle \beta_n(t_n) \rangle$  converges to  $\beta(t)$ .

Theorem 32 also states that the inverse basepoint changing isomorphism is jointly continuous for translation structures on the disk. With small modifications, the above argument proves that if  $\langle t'_n \in S'_n \rangle$  converges to  $\langle t' \in S' \rangle$ , then  $\langle \beta_n^{-1}(t'_n) \in S_n \rangle$  converges to  $\langle \beta^{-1}(t') \in S \rangle$ .  $\square$

## 12. AFFINE ACTIONS

The  $GL(2, \mathbb{R})$  actions behave naturally with respect to universal covering maps. Let  $v : \mathcal{M} \rightarrow \tilde{\mathcal{M}}$  be the map which sends  $S \in \mathcal{M}$  to its universal cover  $\tilde{S} \in \tilde{\mathcal{M}}$ . The action of any  $A \in GL(2, \mathbb{R})$  on  $\mathcal{M}$  satisfies the following commutative diagram:

$$(11) \quad \begin{array}{ccc} \mathcal{M} & \xrightarrow{A} & \mathcal{M} \\ \downarrow v & & \downarrow v \\ \tilde{\mathcal{M}} & \xrightarrow{A} & \tilde{\mathcal{M}} \end{array}$$

Also recall that  $\mathcal{P}$  is the domain of the covering projection and consists of those pair  $(S, \tilde{s})$  with  $\tilde{s} \in \tilde{S}$ . The covering projection  $p : \mathcal{P} \rightarrow \mathcal{E}$  sends  $(S, \tilde{s})$  to its image  $p_S(\tilde{s})$  under the covering map  $p_S : \tilde{S} \rightarrow S$ . Because of the diagram above, the  $GL(2, \mathbb{R})$  actions on  $\mathcal{M}$  and  $\mathcal{E}$  induce an action on  $\mathcal{P}$ . We have the following commutative diagram:

$$(12) \quad \begin{array}{ccc} \mathcal{P} & \xrightarrow{A} & \mathcal{P} \\ \downarrow p & & \downarrow p \\ \mathcal{E} & \xrightarrow{A} & \mathcal{E} \end{array}$$

*Proof of Theorem 12.* Let  $\langle A_n \in GL(2, \mathbb{R}) \rangle$  be a sequence tending to  $A \in GL(2, \mathbb{R})$ . Let  $\langle (S_n, s_n) \in \mathcal{E} \rangle$  be a sequence tending to  $(S, s) \in \mathcal{E}$ . We will show that  $\langle S'_n = A_n(S_n) \rangle$  converges to  $S' = A(S)$  in  $\mathcal{M}$ , and  $\langle s'_n = A_n(s_n) \rangle$  tends to  $s' = A(s)$  in  $\mathcal{E}$ .

First we address convergence of  $\langle S'_n \rangle$  to  $S$ . By Theorem 17 of [Hoo13a], we know that the  $GL(2, \mathbb{R})$  action on  $\tilde{\mathcal{M}}$  is continuous. Then using the continuity of  $v : \mathcal{M} \rightarrow \tilde{\mathcal{M}}$  and commutativity provided by equation 11, we see that the universal covers  $\langle \tilde{S}'_n \rangle$  converge to  $\tilde{S}'$  in  $\tilde{\mathcal{M}}$ .

To show  $\langle S'_n \rangle$  to  $S$ , we will use the convergence criterion of Theorem 17. Consider statement (1). We will use  $p_n, p'_n, p$  and  $p'$  for universal covering maps associated to  $S_n, S'_n, S$  and  $S'$ , respectively. Let  $\tilde{o}' \in p'^{-1}(o_{S'}) \subset \tilde{S}'$ . Let  $\tilde{o} = A^{-1}(\tilde{o}') \in \tilde{S}$ . Then by the commutative diagram given in equation 12,  $\tilde{o} \in p^{-1}(o_S)$ . So by Theorem 17, there is a sequence of points  $\langle \tilde{o}_n \in p_n^{-1}(o_{S_n}) \rangle$  converging to  $\tilde{o}$ . By Theorem 17 of [Hoo13a], the  $GL(2, \mathbb{R})$  action on  $\tilde{\mathcal{E}}$  is continuous. Therefore,  $\langle \tilde{o}'_n = A_n(\tilde{o}_n) \rangle$  converges to  $\tilde{o}'$ . Further, by commutativity we have that  $p'_n(\tilde{o}'_n)$  is the basepoint of  $S'_n$ . This verifies statement (1).

Now we verify statement (2) of Theorem 17. Fix an increasing sequence of positive integers  $\langle n_k \rangle$ . We will abuse notation by using  $k$  to abbreviate  $n_k$ . Suppose that there is a sequence  $\langle \tilde{o}'_k \in p'^{-1}(o_{S'}) \rangle$  which converges to some point  $\tilde{o}' \in \tilde{\mathcal{E}}$ . We must show that  $p'(\tilde{o}') = o_{S'}$ . Since  $\langle \tilde{S}'_k \rangle$  converges to  $\tilde{S}'$ , we know that  $\tilde{o}' \in \tilde{S}'$ . By continuity of the affine action on  $\tilde{\mathcal{E}}$ , we know that  $\langle \tilde{o}_k = A_k^{-1}(\tilde{o}'_k) \rangle$  converges to  $\tilde{o} = A^{-1}(\tilde{o}')$ . By commutative diagram 12, we know that  $p_k(\tilde{o}_k) = o_{S_k}$ . So, by Theorem 17 applied to convergence of  $\langle S_k \rangle$  to  $S$ , we know that  $p(\tilde{o}) = o_S$ . Then again by commutativity, we see that  $p'(\tilde{o}') = A(o_S) = o_{S'}$ . This verifies statement (2), and concludes the proof that  $\langle S'_n \rangle$  converges to  $S$ .

It remains to show that  $\langle s'_n \rangle$  converges to  $s'$ . For this we would use Theorem 18. Statement (1) of this theorem has already been proved above. The second statement is proved in an almost identical way to the proof of criterion (1) of Theorem 17 given two paragraphs above.  $\square$

### 13. THE COMPACTNESS THEOREM

*Proof of Theorem 14.* Since the space  $\mathcal{M}$  is second-countable, sequential compactness is equivalent to compactness. Let  $\langle S_n \rangle$  be a sequence in  $\mathcal{M} \setminus \mathcal{M}_{\neq}(U)$ . We will find a convergent subsequence. We note that it suffices to consider the case when  $U$  is isometric to an open Euclidean metric ball with arbitrary radius  $\epsilon > 0$  and basepoint at its center.

Consider the map  $v : \mathcal{M} \rightarrow \tilde{\mathcal{M}}$  which sends a surface to its universal cover. This map is continuous and

$$v(\mathcal{M} \setminus \mathcal{M}_{\neq}(U)) = \tilde{\mathcal{M}} \setminus \tilde{\mathcal{M}}_{\neq}(U).$$

Note that  $\tilde{\mathcal{M}} \setminus \tilde{\mathcal{M}}_{\neq}(U)$  is compact by Theorem 15 of [Hoo13a]. Since  $\tilde{\mathcal{M}} \setminus \tilde{\mathcal{M}}_{\neq}(U) \subset \tilde{\mathcal{M}} \setminus \tilde{\mathcal{M}}_{\neq}(U)$ , this set is also compact. Thus, we can extract a convergent subsequence of  $\langle \tilde{S}_n \rangle$  which converges to some surface  $\tilde{S} \in \tilde{\mathcal{M}} \setminus \tilde{\mathcal{M}}_{\neq}(U)$ . We can assume by passing to such a subsequence that  $\langle \tilde{S}_n \rangle$  converges to  $\tilde{S}$ .

We will explicitly find a convergent subsequence, which we will describe as an algorithm. The algorithm can be interpreted as an inductive sequence of definitions. In order to describe the algorithm, we first pick a sequence of open metric balls of radius less than  $\frac{\epsilon}{2}$ ,  $\langle B_i \subset \tilde{S} \rangle_{i \in \mathbb{N}}$ , so that  $\tilde{S} = \bigcup_{i \in \mathbb{N}} B_i$ . The purpose of the algorithm is to find a convergent subsequence  $\langle S_{n_k} \rangle$  of  $\langle S_n \rangle$ . So, we will define an increasing sequence of natural numbers  $\langle n_k \rangle$ . We will also construct a subset  $\mathcal{I} \subset \mathbb{N}$  and a collection  $\tilde{O} = \{\tilde{o}_i \in \tilde{S} : i \in \mathcal{I}\}$  indexed with repeats. The collection  $\tilde{O} \subset \tilde{S}$  will later be the collection of lifts of the basepoint of the limiting surface  $S$ . After describing the algorithm, we will prove several statements about the objects produced, and define  $S$ .

We now specify some notation. We let  $p_n : \tilde{S}_n \rightarrow S_n$  denote the universal covering maps. We denote the basepoint of  $S_n$  by  $o_n$ .

The following is the aforementioned algorithm:

- (1) Set  $N_0 = \mathbb{N} = \{1, 2, 3, \dots\}$ .
- (2) Set  $\mathcal{K} = \emptyset \subset \mathbb{N}$ .
- (3) Set  $\tilde{O} = \emptyset \subset \tilde{S}$ .
- (4) Evaluate the following statements for each integer  $k \geq 1$  in order of increasing  $k$ :
  - (a) If there is an increasing sequence of integers  $\langle m_j = m_j(k) \in N_{k-1} : j \in \mathbb{N} \rangle$  and a sequence  $\langle \tilde{o}_{m_j}^k \in p_{m_j}^{-1}(o_{m_j}) \rangle$  which converges to a point  $\tilde{o}^k \in B_k$ , then choose such a sequence and evaluate the following:
    - (i) Add  $\tilde{o}^k$  to the set  $\tilde{O} \subset \tilde{S}$ .
    - (ii) Add  $k$  to the set  $\mathcal{K} \subset \mathbb{N}$ .
    - (iii) Define  $n_k = m_1(k) = \min\{m_j(k) : j \in \mathbb{N}\}$ .
    - (iv) Define  $N_k = \{m_j(k) : j \in \mathbb{N}\} \setminus \{n_k\} \subset N_{k-1}$ .
  - (b) Otherwise (if there are no such sequences  $\langle m_j \rangle$  and  $\langle \tilde{o}_{n_j}^k \rangle$ ), evaluate the following statements:
    - (v) Define  $n_k = \min N_{k-1}$ .
    - (vi) Define  $N_k = N_{k-1} \setminus \{n_k\}$ .

We will show that the subsequence  $\langle S_{n_k} \rangle$  converges by using the criterion from Theorem 17 applied to the constructed subset  $\tilde{O} = \{\tilde{o}^k : k \in \mathcal{K}\}$ .

We now consider the first statement of Theorem 17. Choose a  $\tilde{o}^i \in \tilde{O}$ . Then  $i \in \mathcal{K}$ , and there is a sequence  $\langle m_j(i) \rangle$  and a choice of  $\langle \tilde{o}_{m_j(i)}^i \in p_{m_j}^{-1}(o_{m_j(i)}^i) \rangle$  which converges to  $\tilde{o}^i$ . By construction, our constructed sequence  $\langle n_k : k \geq i \rangle$  lies in  $N_k$  and thus is a subsequence of  $\langle m_j(i) : j \in \mathbb{N} \rangle$ . In particular,  $\langle \tilde{o}_{n_k}^i : k \geq i \rangle$  converges to  $\tilde{o}^i$ . This proves that statement (1) of Theorem 17 holds.

Now we turn our attention to statement (2). Suppose that  $\langle k(\ell) : \ell \in \mathbb{N} \rangle$  is an increasing sequence in  $\mathbb{N}$ , and that there is a sequence  $\langle \tilde{o}_{n_{k(\ell)}} \rangle$  which converges in  $\tilde{\mathcal{E}}$ . Let  $\tilde{o}$  be the limit point, which must lie in  $\tilde{S}$ . We must show that  $\tilde{o} \in \tilde{O}$ . Since  $\{B_i\}$  was a covering of  $\tilde{S}$ , there is an  $i$  so that  $\tilde{o} \in B_i$ . As above, we note that  $\langle n_k : k \geq i \rangle$  is a subsequence of  $N_{k-1}$ . In particular, there is a subsequence of  $N_{k-1}$  which converges to a point in  $B_i$ . So, in step (a), we must have chosen a sequence  $\langle m_j = m_j(i) \in N_{j-1} \rangle$  and a sequence  $\langle \tilde{o}_{m_j}^i \rangle$  which converges to a  $\tilde{o}^i \in B_i$ . We will show that  $\tilde{o} = \tilde{o}^i$ , which will prove that  $\tilde{o} \in \tilde{O}$ . Again note that  $\langle n_k : k \geq i \rangle$  is a subsequence of  $\langle m_j \rangle$ . The sequence  $\langle n_{k(\ell)} : \ell \in \mathbb{N} \rangle$  is a further subsequence. Therefore, we have

$$\lim_{\ell \rightarrow \infty} \tilde{o}_{n_{k(\ell)}} = \tilde{o} \quad \text{and} \quad \lim_{\ell \rightarrow \infty} \tilde{o}_{n_{k(\ell)}}^i = \tilde{o}^i.$$

Assume that  $\tilde{o} \neq \tilde{o}^i$ . Choose a  $D \in \overline{\text{Disk}}(\tilde{S})$  so that  $B_i \subset D^\circ$ . Since the limit points are distinct, we can choose disjoint open sets  $U \subset B_i$  and  $V \subset B_i$  so that  $\tilde{o} \in U$  and  $\tilde{o}^i \in V$ . Since  $\langle \tilde{S}_{n_{k(\ell)}} \rangle$  converges to  $S$ , there is an  $L_1$  so that for  $\ell > L_1$ , we have an immersion  $\iota_\ell : D \rightsquigarrow \tilde{S}_{n_{k(\ell)}}$ . Now observe that  $\tilde{o} \in \mathcal{E}_+(D, U)$ . Therefore, there is an  $L_2 > L_1$  so that for  $\ell > L_2$ , we have  $\tilde{o}_{n_{k(\ell)}} \in \iota_\ell(U)$ . Similarly, there is an  $L_3 > L_2$  so that for  $\ell > L_3$ , we have  $\tilde{o}_{n_{k(\ell)}}^i \in \iota_\ell(V)$ . In particular,

$$\tilde{o}_{n_{k(\ell)}}, \tilde{o}_{n_{k(\ell)}}^i \in \iota_\ell(B_i) \quad \text{for } \ell > L_3.$$

Since  $B_i$  is a ball of radius less than  $\frac{\epsilon}{2}$ , these points are distance less than  $\epsilon$  apart in  $\tilde{S}_{n_{k(\ell)}}$ . But, since  $U$ , a Euclidean open ball of radius  $\epsilon$ , embeds into  $\tilde{S}_{n_{k(\ell)}}$  about the basepoint, it must be that these points are equal when  $\ell > L_3$ . Therefore the limits of these sequences are the same, and  $\tilde{o} = \tilde{o}^i \in \tilde{O}$ .  $\square$

## REFERENCES

- [Bow13] Joshua P. Bowman, *The complete family of Arnoux-Yoccoz surfaces*, *Geometriae Dedicata* **164** (2013), no. 1, 113–130 (English).
- [BV13] Joshua P. Bowman and Ferrán Valdez, *Wild singularities of flat surfaces*, *Israel J. Math.* **197** (2013), no. 1, 69–97. MR 3096607
- [Del13] Vincent Delecroix, *Divergent trajectories in the periodic wind-tree model.*, *J. Mod. Dyn.* **7** (2013), no. 1, 1–29 (English).
- [DHL11] Vincent Delecroix, Pascal Hubert, and Samuel Lelièvre, *Diffusion for the periodic wind-tree model*, preprint <http://arxiv.org/abs/1107.1810>, 2011.
- [FU11] Krzysztof Frączek and Corinna Ulcigrai, *Non-ergodic Z-periodic billiards and infinite translation surfaces*, preprint <http://arxiv.org/abs/1109.4584>, 2011.
- [FU12] Krzysztof Frączek and Corinna Ulcigrai, *Ergodic directions for billiards in a strip with periodically located obstacles*, 2012.
- [HHW13] W. Patrick Hooper, Pascal Hubert, and Barak Weiss, *Dynamics on the infinite staircase*, *Discrete and Continuous Dynamical Systems - Series A* **33** (2013), no. 9, 4341–4347.



- [HLT11] Pascal Hubert, Samuel Lelièvre, and Serge Troubetzkoy, *The Ehrenfest wind-tree model: Periodic directions, recurrence, diffusion.*, J. Reine Angew. Math. **656** (2011), 223–244 (English).
- [Hoo10] W. Patrick Hooper, *The invariant measures of some infinite interval exchange maps*, preprint, <http://arxiv.org/abs/1005.1902>, 2010.
- [Hoo13a] W. Patrick Hooper, *Immersion and translation structures on the disk*, 2013, preprint, [arXiv:1309.4795](http://arxiv.org/abs/1309.4795).
- [Hoo13b] ———, *An infinite surface with the lattice property I: Veech groups and coding geodesics*, to appear in Transactions of the AMS, <http://arxiv.org/abs/1011.0700>, 2013.
- [HS10] Pascal Hubert and Gabriela Schmithüsen, *Infinite translation surfaces with infinitely generated Veech groups*, J. Mod. Dyn. **4** (2010), no. 4, 715–732. MR 2753950 (2012e:37075)
- [HW12] W. Patrick Hooper and Barak Weiss, *Generalized staircases: recurrence and symmetry.*, Ann. Inst. Fourier **62** (2012), no. 4, 1581–1600 (English. French summary).
- [HW13] Pascal Hubert and Barak Weiss, *Ergodicity for infinite periodic translation surfaces*, Compos. Math. **149** (2013), no. 8, 1364–1380 (English).
- [MT02] Howard Masur and Serge Tabachnikov, *Rational billiards and flat structures*, Handbook of dynamical systems, Vol. 1A, North-Holland, Amsterdam, 2002, pp. 1015–1089. MR 1928530 (2003j:37002)
- [PSV11] Piotr Przytycki, Gabriela Schmithüsen, and Ferrán Valdez, *Veech groups of Loch Ness monsters.*, Ann. Inst. Fourier **61** (2011), no. 2, 673–687 (English).
- [RT12] David Ralston and Serge Troubetzkoy, *Ergodic infinite group extensions of geodesic flows on translation surfaces*, Journal of Modern Dynamics **6** (2012), no. 4, 477–497.
- [RT13] ———, *Ergodicity of certain cocycles over certain interval exchanges*, Discrete and Continuous Dynamical Systems **33** (2013), 2523–2529.
- [Sch11] Martin Schmoll, *Veech groups for holonomy free torus covers*, Journal of Topology and Analysis (2011), to appear.
- [Tab05] Serge Tabachnikov, *Geometry and billiards.*, Providence, RI: American Mathematical Society (AMS), 2005 (English).
- [Thu97] William P. Thurston, *Three-dimensional geometry and topology. Vol. 1*, Princeton Mathematical Series, vol. 35, Princeton University Press, Princeton, NJ, 1997, Edited by Silvio Levy. MR 1435975 (97m:57016)
- [Tre12] Rodrigo Treviño, *On the ergodicity of flat surfaces of finite area*, 2012, preprint, <http://arxiv.org/abs/1211.1313>.
- [Tro10] Serge Troubetzkoy, *Typical recurrence for the Ehrenfest wind-tree model.*, J. Stat. Phys. **141** (2010), no. 1, 60–67 (English).
- [Vee89] W. A. Veech, *Teichmüller curves in moduli space, Eisenstein series and an application to triangular billiards*, Invent. Math. **97** (1989), no. 3, 553–583. MR 1005006 (91h:58083a)
- [ZK75] A. N. Zemljakov and A. B. Katok, *Topological transitivity of billiards in polygons*, Mat. Zametki **18** (1975), no. 2, 291–300. MR 0399423 (53 #3267)
- [Zor06] Anton Zorich, *Flat surfaces*, Frontiers in number theory, physics, and geometry. I, Springer, Berlin, 2006, pp. 437–583. MR 2261104 (2007i:37070)
- [Zvo12] Dimitri Zvonkine, *An introduction to moduli spaces of curves and their intersection theory*, Handbook of Teichmüller theory. Volume III, IRMA Lect. Math. Theor. Phys., vol. 17, Eur. Math. Soc., Zürich, 2012, pp. 667–716. MR 2952773

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