DYNAMICS ON THE INFINITE STAIRCASE

W. PATRICK HOOPER, PASCAL HUBERT, AND BARAK WEISS

Abstract. For the 'infinite staircase' square tiled surface we classify the Radon invariant measures for the straight line flow, obtaining an analogue of the celebrated Veech dichotomy for an infinite genus lattice surface. The ergodic Radon measures arise from Lebesgue measure on a one parameter family of deformations of the surface. The staircase is a $\mathbb{Z}$-cover of the torus, reducing the question to the well-studied cylinder map.

Gutkin [G] and Veech [V1, V2] classified the invariant measures for a linear flow on a square tiled surface (of finite area), showing that it satisfies the Veech dichotomy: the flow is uniquely ergodic in any direction with irrational slope and periodic in any rational direction. Motivated by recent works on infinite genus translation surfaces (see e.g. [H1], [Va]) we consider the dynamics of the linear flow on an *infinite* square tiled surface. Employing results of [ANSS], we obtain a complete classification of invariant Radon measures for a particular infinite surface, the 'infinite staircase' surface.

This surface, which we denote by M, is an infinite polygon with identifications on the boundaries (see figure 1). Let R be the rectangle $[0, 2] \times [0, 1]$, and consider the following boundary identifications on $R \times \mathbb{Z}$: for any $k \in \mathbb{Z}$, opposing vertical sides of $R \times \{k\}$ are glued to each other. The left (resp. right) half of the top boundary of $R \times \{k\}$ is glued to the right (resp. left) half of the bottom of $R \times \{k-1\}$ (resp. $R \times \{k+1\}$). Thus we will write points of M as (x, k) where $x \in R$ and $k \in \mathbb{Z}$.

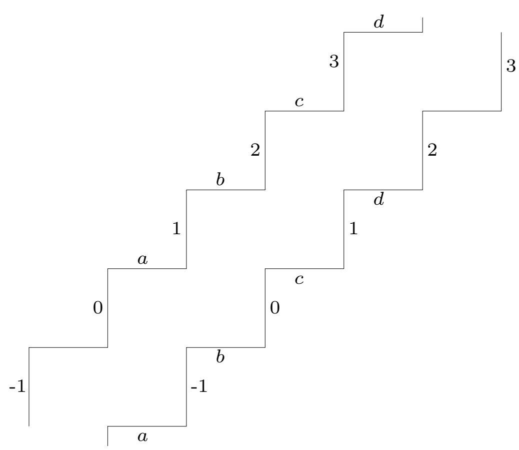

Figure 1. The infinite staircase

Removing from M the points (x, k) for which $x \in R$ has integral coordinates, we obtain an infinite genus orientable surface M', endowed with a complex structure and holomorphic 1-form. This surface is an infinite square tiled surface. M is the completion of M' with respect to the flat metric, and $M \setminus M'$ consists of four cone singularities with "infinite cone angle." There is an obvious $\mathbb{Z}$-action on M by translations, we call this the *deck group* action since it corresponds to the action of the deck group in the covering map from M' to the twice-punctured torus. Valdez pointed out to us that M' is homeomorphic to a 'Loch Ness monster' (see [Va] for a definition). Denote by ϕ_s^α the flow in direction (of slope) α to time s. We use the same notation for the flow on both M and $\mathbb{T}^2$.

In this note, we use results about cylinder maps (skew products over rotations) to prove the following analogue of the Veech dichotomy:

Theorem 1. (i) *If α is rational of the form p/q with p, q coprime and p or q even, then, in direction of slope α, the surface M decomposes into an infinite number of periodic cylinders.*
(ii) *If α is rational of the form p/q with p, q coprime and p and q both odd, then, in direction of slope α, the surface decomposes into two infinite strips.*
(iii) *For every irrational α, the flow in direction of slope α is ergodic with respect to Lebesgue measure.*
(iv) *For every irrational α, the locally finite Borel ergodic measures for the flow in direction of slope α are precisely the Maharam measures.*

The Maharam measures mentioned in case (iv) are obtained from a general construction valid for any G-valued skew product over a probability measure preserving transformation, where G is a locally compact abelian group; see [M] and [ANSS] for definitions. In our context, by the results of [ANSS], for each irrational α and each positive real parameter η, there is (up to scaling) one such measure $\mu = \mu_\alpha^\eta$; it is characterized by the formula

$$d\mu(x,k) = e^{-k\eta} dm(x), \tag{1}$$

where m is a measure on the quotient of M by the deck group which is quasi-invariant under the flow ϕ_s^α and satisfies

$$\frac{d\phi^\alpha_{s_0 *} m}{dm}(x) = e^\eta, \tag{2}$$

where s_0 is the return time to the bottom of R.

The following provides a geometric interpretation of the Maharam measures. Recall that for any translation surface M_0 and any slope α, there is a foliation $\mathcal{F}_\alpha$ of M_0 by lines of slope α. The leaves of this foliation are orientable and the ϕ^α-invariant Radon measures on M_0 are in one-to-one correspondence with transverse measures on M_0 to $\mathcal{F}_\alpha$ (this correspondence will be recalled in detail below).

Theorem 2. *Given any $\eta \in \mathbb{R}$, there is a translation surface M_η (obtained by deforming M), so that for each irrational α there is $\alpha' = \alpha'(\alpha, \eta)$ and a continuous surjective map $H = H(\eta, \alpha)$ from M_η to M such that*
a) *The pushforward of the foliation $\mathcal{F}_{\alpha'}$ on M_η under H is the foliation $\mathcal{F}_\alpha$ on M.*
b) *The transverse measure corresponding to μ_α^η is the pushforward under H of the transverse measure corresponding to Lebesgue measure on M_η.*

Proof of Theorem 1. Rational directions. An affine automorphism of M is a diffeomorphism of M that acts as a matrix of $\mathrm{SL}_2(\mathbb{R})$ up to cutting M into polygons and repasting. Its image in $\mathrm{SL}_2(\mathbb{R})$ under the map sending an affine map to its linear part is called the *Veech group* of M, and denoted by Γ. It is easy to see that Γ is a lattice in $\mathrm{SL}_2(\mathbb{R})$, in fact

$$\Gamma = \left\langle \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\rangle,$$

a 2-cusp subgroup of $\mathrm{SL}_2(\mathbb{Z})$ of index 3. In particular its set of cusp directions is $P^1(\mathbb{Q})$. It is classical that one cusp corresponds to rational numbers p/q with p and q odd. This is the orbit of 1. The other cusp corresponds to the orbit of 0: the rational numbers p/q with p or q even and the point at infinity.

Note that an affine automorphism with linear part A intertwines the linear flow in direction α with the linear flow in direction $A\alpha$ (where A acts on directions by projectivizing the linear action). Since in the horizontal direction, the surface is decomposed into an infinite number of cylinders (of finite area), the same is true in the orbit of 0 by the Veech group. In the direction of slope 1, the surface M is decomposed into two infinite strips of the same width. This proves (i) and (ii).

Irrational directions. We will reduce the proofs of (iii), (iv) to known results on skew products over rotations. Recall that if $\alpha \in \mathbb{R}$ and $f : \mathbb{T}^1 \to \mathbb{Z}$ is measurable, then the *cylinder map* over the rotation $R_\alpha : x \to x + \alpha \mod 1$ is the map

$$T_{\alpha,f} : \mathbb{T}^1 \times \mathbb{Z} \to \mathbb{T}^1 \times \mathbb{Z}, \quad (x,k) \mapsto (R_\alpha(x), k + f(x))$$

The map f is called the *cocycle* of the cylinder map.

Recall that $\mathrm{GL}_2(\mathbb{R})$ acts on translation surfaces via its linear action on the plane. We change M by applying to it the map $h = \begin{pmatrix} 1/2 & -1/2 \\ 0 & 1 \end{pmatrix}$, which is a horizontal shear mapping the direction of slope 1 to the vertical direction, followed by a contraction of the horizontal direction by a factor of 1/2. In particular h maps R to a parallelogram R' which can be identified with the unit square by considering coordinates mod 1, and maps irrational directions to themselves, so that instead of studying the flows on ϕ_s^α we may study their conjugates under h. With this new parametrization:

- Points $m \in M$ are of the form (x,k) where $x \in R'$ and $k \in \mathbb{Z}$. We call $k = k(m)$ the *level* of m.
- For every k, the vertical boundaries of $R' \times \{k\}$ are glued to each other by a horizontal translation.
- Denoting by I the top boundary of R' and by J its bottom boundary, we have that I is partitioned into intervals $I_1 = [0, 1/2), I_2 = [1/2, 1)$, and J is partitioned into $J_1 = [0, 1/2), J_2 = [1/2, 1)$, and for each $k \in \mathbb{Z}$, $I_1 \times \{k\}$ is glued by translation to $J_1 \times \{k+1\}$ and $I_2 \times \{k\}$ is glued by translation to $J_2 \times \{k-1\}$.
- The first return map of the flow of slope α on $I \times \mathbb{Z}$ is the cylinder map T_{α,f_0} over the rotation R_α, where the cocycle is the step function

$$f_0 : I \to I, \quad x \mapsto (-1)^{j+1} \text{ for } x \in I_j, \ j = 1, 2.$$

Since $I \times \mathbb{Z}$ intersects all orbits infinitely often with a constant return time, we have a bijection between the invariant Radon measures for the flow ϕ_s^α and those for T_{α,f_0}. The ergodicity of Lebesgue measure for the cylinder map was proven by Conze [Co]. This implies (iii). To prove (iv) we apply a result of Aaronson, Nakada,

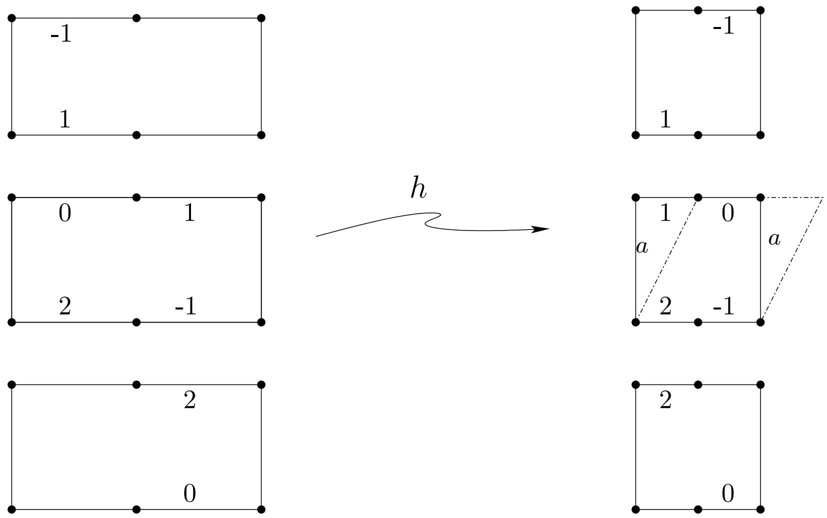

FIGURE 2. Changing the staircase to a cylinder flow

Sarig and Solomyak [ANSS, Theorem 2.4] showing that the locally finite ergodic invariant measures for the cylinder flow are precisely the Maharam measures. □

Proof of Theorem 2. For each $\eta \in \mathbb{R}$, construct a 'deformed staircase' M_η as follows. Let $t = e^{\eta/2}$, and for each $k \in \mathbb{Z}$, let R_k be a rectangle in the plane, with sides parallel to the coordinate axes, with vertical side of length t^{2k}, and horizontal side of length $t^{2k-1}+t^{2k+1}$. Now paste the right-hand (respectively left-hand) portion of the top edge of R_k, of length t^{2k+1} (resp. t^{2k-1}), to the left-hand (resp. right-hand) portion of the bottom edge of R_{k+1} (resp. R_{k-1}). See Figure 3.

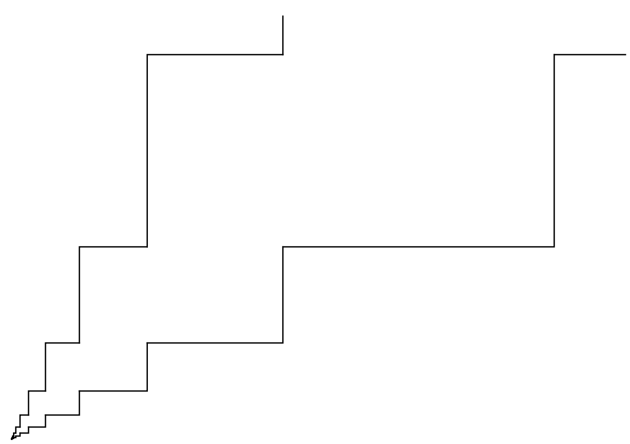

FIGURE 3. The deformed staircase M_η, with $e^\eta = 2$

The surface M_η has been constructed so that the group $\Gamma_\eta = \langle h_{t'}, n_{t'} \rangle$ is contained in its Veech group, where

$$t' = t'(\eta) = t + 1/t, \quad h_s = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad n_s = \begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix}.$$

Indeed, in the horizontal and vertical directions, the surface decomposes into horizontal and vertical cylinders each of inverse-modulus equal to t'. The group homomorphism $\Gamma_\eta \to \Gamma$ defined by $h_{t'} \mapsto h_2,\ n_{t'} \mapsto n_2$ identifies Γ_η with an isomorphic index two subgroup of Γ, and these two groups act in the same way on corresponding cylinders. In fact, for all $\eta \neq 0$, the group Γ_η is the Veech group of M_η, but we will not be using this.

Now let $\widehat{M_\eta}$ be the topological space obtained by pasting the right-hand (respectively left-hand) portion of the top edge of R_0, of length t (resp. t^{-1}), to the left-hand (resp. right-hand) portion of the bottom edge of R_0, by a dilation by a factor of t^{-2} (resp. t^2). The group $\mathbb{Z}$ acts on M_η by moving up and down along levels and expanding, and $\widehat{M_\eta}$ is the quotient of M_η by this action. It is topologically a torus, endowed with the structure of a *dilation surface*, i.e. equipped with an atlas of charts into the plane whose transition maps are compositions of translations and homotheties. This atlas is defined away from a pair of singularities. See figure 4. Note that like a translation surface, a dilation surface has foliations $\mathcal{F}_\theta$ by parallel lines of slope θ.

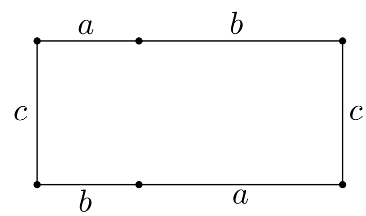

FIGURE 4. The dilation surface $\widehat{M_\eta}$

Fix $\sigma \subset \widehat{M_\eta}$ to be the bottom segment of the rectangle R_0, and consider the map $\tau = \tau(\alpha') : \sigma \to \sigma$ obtained by moving in $\widehat{M_\eta}$ along leaves of $\mathcal{F}_{\alpha'}$. This is a circle homeomorphism, so by classical results (see e.g. [HK, §11.1]) is semi-conjugate[1] to a unique rotation $t \mapsto t + \alpha$, and $\alpha \in \mathbb{R}/\mathbb{Z}$ is called the rotation number of the homeomorphism. We will require the following:

Claim 3. *For each irrational α, there is α' such that $\tau(\alpha')$ has rotation number α.*

The claim will be proved further below; assuming its validity, let $R_\alpha : I \to I$ be the rotation by α, which, as in the proof of Theorem 1, we consider as the return map along the leaves of $\mathcal{F}_\alpha$ to a transverse segment I in the quotient torus $R' = M/\mathbb{Z}$. We will take this segment $I = [0, 2]$ to be the bottom edge of the rectangle R mentioned in the second paragraph of this note. By the above, there is a semi-conjugacy $\hat{H} : \sigma \to I$; that is $\hat{H} \circ \tau = R_\alpha \circ \hat{H}$.

We are free to choose this semi-conjugacy up to post composition with a rotation, and so we choose $\hat{H}$ so that it takes the left endpoint of σ to the left endpoint of I.

[1]Since τ is piecewise affine, results of Herman [He] actually imply that it is *conjugate* to a rotation, but we will not need this.

Let $\sigma_1, \sigma_2 \subset \sigma$ be the two horizontal segments joining the singularities. We claim that $\hat{H}$ also takes the second singularity of $\widehat{M_\eta}$ to the midpoint of I. To see this, note that the surface $\widehat{M_\eta}$ admits a hyperelliptic involution which swaps σ_1 and σ_2 and preserves the dilation surface structure (thinking of $\widehat{M_\eta}$ as two rectangles with horizontal sides glued to each other, the involution rotates each rectangle by an angle π around its midpoint). Let $\iota : \sigma \to \sigma$ be the restriction of this involution to σ. Because ι swaps σ_1 and σ_2 and satisfies $\iota \circ \tau \circ \iota = \tau^{-1}$, the ergodic averages of the characteristic functions of these two intervals are equal. So $\hat{H}$ must take each interval σ_i to an interval which is half the length of I, and must take the second singularity to the midpoint of I.

We can then extend $\hat{H}$ to a continuous surjective map $H : M_\eta \to M$ as follows. Let σ_k be the component of the pre-image of σ, contained in the bottom edge of R_k and let J_k be the corresponding segment in M. There is a unique way to define H on each σ_k so that $H(\sigma_k) = J_k$ and H is a lift of $\hat{H}$. Now extend H to a continuous map $M_\eta \to M$ by requiring that it map the foliation $\mathcal{F}_{\alpha'}$ on M_η to $\mathcal{F}_\alpha$ on M.

Our construction ensures that property a) is satisfied, and we proceed to show b). We recall the correspondence between straightline flow invariant Radon measures and transverse measures to the foliation by straight lines. A transverse measure to a foliation $\mathcal{F}$ is an assignment, for each compact arc δ transverse to $\mathcal{F}$, of a finite Borel measure μ_δ, such that the system of measures is invariant under isotopy through arcs transverse to $\mathcal{F}$. The relation between a flow-invariant measure and the system of transverse measures is given by the formula

$$\varepsilon\, \mu_\delta(A) = \mu\left(\bigcup_{0<s<\varepsilon} \phi_s(A)\right),$$

for any $0 < \varepsilon < \varepsilon_0 = \varepsilon_0(\delta)$. Here ϕ_s denotes the straightline flow.

It follows from the description above, that on each σ_k that the transverse measure associated to Lebesgue measure on M_η is simply the length element dx, multiplied by a constant depending only on α (and independent of k). Let μ_k denote the restriction of this length measure to each σ_k and let $m = \hat{H}_*\mu_0$ be the pushforward of the measure defined on σ_0. Since the deck group acts by dilations on M_η, the choice of t ensures that the pushforward of μ_k under H scales by a factor of $e^{-k\eta}$. That is, (1) and (2) hold. It is shown in [ANSS] that these properties characterize the Maharam measure up to scaling. Thus b) holds. □

Proof of Claim 3. When we decrease α', the maps $\tau(\alpha')$ increase in the following sense. For each α', let $F_{\alpha'} : \mathbb{R} \to \mathbb{R}$ be a lift of $\tau(\alpha')$ (where $\mathbb{R}$ is thought of as the universal cover of S^1). Such a lift is well-defined up to pre- and post-composition with adding an integer. We can choose the lifts so that $F_{\alpha'}(x)$ varies continuously with α' for each x. Then for any $\alpha'' < \alpha'$ and any x we have $F_{\alpha''}(x) > F_{\alpha'}(x)$. By [HK, Prop. 11.1.8-9], the rotation number of $\tau(\alpha')$ is a decreasing function of α' which is strictly decreasing at irrational points, and by [HK, Prop. 11.1.6] this map is continuous. We find that the map $S^1 \to S^1$ which assigns to α' the rotation number of $\tau(\alpha')$ is non-constant, continuous, and monotone. Thus it is surjective. □

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Department of Mathematics, The City College of New York, NAC 8/133, Convent Ave at 138th Street, New York, NY, USA 10031 `whooper@ccny.cuny.edu`

LATP, case cour A, Faculté des sciences de Saint Jérôme, Avenue Escadrille Normandie-Niemen, 13397 Marseille Cedex 20, France

`hubert@cmi.univ-mrs.fr`

Ben Gurion University, Be'er Sheva, Israel 84105 `barakw@math.bgu.ac.il`