FROM PAPPUS' THEOREM TO THE TWISTED CUBIC

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ABSTRACT. We discuss a classical result in planar projective geometry known as Steiner's theorem involving 12 interlocking applications of Pappus' theorem. We prove this result using three dimensional projective geometry then uncover the dynamics of this construction and relate them to the geometry of the twisted cubic.

INTRODUCTION

Given a pair of lines with three points on each, Pappus' theorem allows us to construct a new line. In fact, these points allow for six distinct applications of Pappus' theorem, constructing six lines such as the ones shown to the right below. Through computer experimentation or simply with pen, paper, and a straight-edge one could discover the beginnings of Steiner's theorem, that these six lines are concurrent in threes.

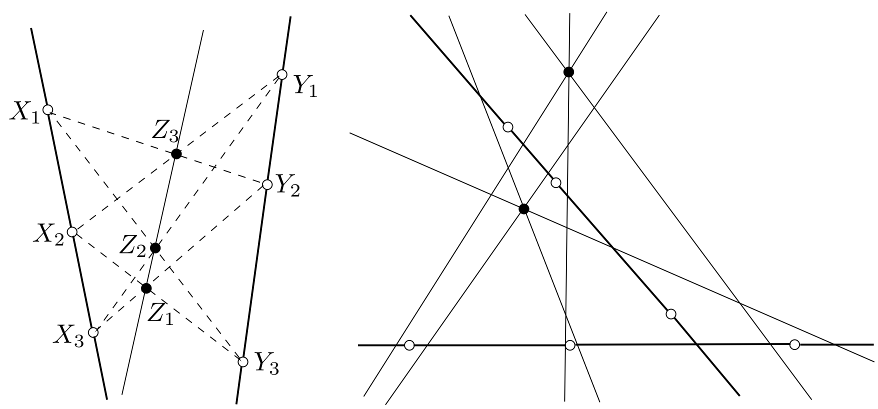

FIGURE 1. Pappus' theorem and the six lines constructible via Pappus' theorem.

Steiner's theorem manifests the surprising symmetry of Pappus' theorem. This paper is an exploration of this symmetry. We provide a proof of Pappus' theorem which opens the door to a close relationship between Pappus' theorem and certain constructions in projective 3-space. We use this viewpoint to prove Steiner's theorem. It should be noted that Steiner's theorem is a known result, but we have not seen an elementary proof of it. The need for such a proof was expressed in [Rig83].

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While the proof of Steiner's theorem is the theme of this paper, probably our deepest result is about a special kind of hexagon in projective 3-space (the skew hexagon of the skew hexagon theorem). This theorem can be viewed as an unlikely marriage between Pappus' theorem and Desargue's theorem. A second view depicts this skew hexagon theorem as a geometric construction describing the *skew-symmetric polarities* of projective 3-space. Skew-symmetric polarities are simply a special kind of duality and will be distinguished from symmetric polarities. We will define and describe these notions in §2.

We state Pappus' theorem, Steiner's theorem, and our results on its dynamics in the next section. We follow this with a short introduction to projective geometry, then we prove Pappus' theorem and draw the connection to projective 3-space. Steiner's theorem is then proved with the aid of the skew hexagon theorem. In the final section, we use the geometry of the twisted cubic to prove results involving the dynamics of Steiner's theorem over algebraically closed fields.

Throughout this paper, we will be concerned with projective geometry over a base field of characteristic not two or three.

1. The statement of Steiner's Theorem and dynamical results

Consider two lines, x_l and y_l, in the projective plane. Let $x = (X_1, X_2, X_3)$ be a triple of distinct points on the line x_l and $y = (Y_1, Y_2, Y_3)$ be a triple of distinct points on y_l.

Theorem 1.1 (Pappus' Theorem)**.** *The points $Z_1 = \overline{X_2Y_3} \cap \overline{X_3Y_2}$, $Z_2 = \overline{X_3Y_1} \cap \overline{X_1Y_3}$, and $Z_3 = \overline{X_1Y_2} \cap \overline{X_2Y_1}$ are colinear. (As in figure 1)*

We define $\ell(x, y) = \overline{Z_1Z_2Z_3}$ to be the map which takes two ordered triples of colinear points to the line constructed by Pappus' Theorem. The permutation group, S_3, acts on triples of points. If $\upsilon \in S_3$ we define the group action

$$\upsilon(Y_1, Y_2, Y_3) = (Y_{\upsilon^{-1}(1)}, Y_{\upsilon^{-1}(2)}, Y_{\upsilon^{-1}(3)}) \tag{1}$$

By inspection it can be seen that:

(1) $\ell(y, x) = \ell(x, y)$
(2) $\ell(\upsilon x, \upsilon y) = \ell(x, y)$

These statements follow from the fact that the construction constructs the same points $\{Z_i\}$, so ℓ must construct the same line. Therefore, the six lines $\ell(x, \upsilon y)$ exhaust the lines constructible with Pappus' theorem from these triples.

Theorem 1.2 (Steiner's Theorem I)**.** *The six lines $\ell(x, \upsilon y)$, $\upsilon \in S_3$ are concurrent in threes. More precisely, the the lines corresponding to even permutations are concurrent. Similarly, the lines corresponding to odd permutations are concurrent.*

For now on, we enumerate S_3 as the group $<\sigma, \tau | \sigma^3 = \tau^2 = (\sigma\tau)^2 = e>$. We label the triples of concurrent lines $z_{even} = (\ell(x, y), \ell(x, \sigma y), \ell(x, \sigma^2 y))$ and $z_{odd} = (\ell(x, \tau y), \ell(x, \tau\sigma y), \ell(x, \tau\sigma^2 y))$.

The projective plane has the nice property that any two points can be joined by a line and any two lines intersect at a point. This notion is generalized by the Principal of Duality, which tells us that any true statement in terms of points and lines has a true dual statement, constructed by swapping the notions of point with line, join with intersection, and colinear with concurrent. In particular there is a dual to Pappus' theorem, which takes as input two triples of concurrent lines and outputs a point. We will denote this dual operation as ℓ^*.

Now consider $\ell^*(z_{even}, \upsilon z_{odd})$ for $\upsilon \in S_3$. By the dual to Steiner's Theorem I, the points constructed are colinear in threes. Surprisingly, the points return to the original lines:

Theorem 1.3 (Steiner's Theorem II)**.** *$\ell^*(z_{even}, \upsilon z_{odd})$ lies on x_l when υ is even and on y_l when υ is odd.*

Let $O = x_l \cap y_l$ be the point colinear with both the triples x and y. Choose a new line y_l' that contains the point O and a triple of points y'. Then we can again apply Pappus' theorem to the triples of points x and y' in the six different ways, again getting new triples of concurrent lines according to Steiner's Theorem I, say z_{even}' and z_{odd}'. Now we construct the points $\ell^*(z_{even}', \upsilon z_{odd}')$ for υ even. The final stage of Steiner's theorem tells us that

$$\ell^*(z_{even}, \sigma^i z_{odd}) = \ell^*(z_{even}', \sigma^i z_{odd}') \quad \forall i = 0, 1, 2$$

That is, the points constructed in Steiner's Theorem II that lie on x_l are identical in both cases.

Theorem 1.4 (Steiner's Theorem III)**.** *The points $\ell^*(z_{even}, \upsilon z_{odd})$ for even permutations υ are dependent on the point $O = x_l \cap y_l$ and the triple x, but independent of the choice of line y_l through O and of the triple of points y.*

We define the *Steiner Map* to be the map

$$\mathcal{S}_O : x \mapsto (\ell^*(z_{even}, z_{odd}), \ell^*(z_{even}, \sigma^2 z_{odd}), \ell^*(z_{even}, \sigma z_{odd}))$$

Where we define z_{even} and z_{odd} as above based on x, O, and a triple of points y colinear with O (the choice of y being necessary for construction but insignificant to the result).

We are interested in the dynamics of this map, but rather than dealing directly with this map, we choose to instead look at a nice conjugate. We will show in corollary 4.8 that permutations commute with $\mathcal{S}_O$, as intuitively, we have constructed a quantity which "considers the permutations of x equally." Thus it makes sense to consider S_O as a map on unordered triples of points. The space of all unordered triples of points in a projective line is canonically isomorphic to projective 3-space, under the projectivization of the elementary symmetric polynomials. We denote this isomorphism as Σ and it is the projectivization of the map (where F denotes the ground field):

$$\Sigma : F \times F \times F \to F^3 : (a, b, c) \mapsto (a + b + c, ab + ac + bc, abc)$$

The map Σ has the property that it takes the space of all triples of identical points to a twisted cubic. In the algebraically closed case, the space of all secants (lines which pass through the twisted cubic at two points) and tangents of the twisted cubic covers projective 3-space. More specifically, for any point in space not on the twisted cubic, there is a unique secant or tangent of the twisted cubic passing through that point.

We are interested in the map $\Sigma \circ \mathcal{S}_O \circ \Sigma^{-1}$. This map turns out to be a homogeneous rational map of degree six which preserves nearly every secant of the twisted cubic, but not the points on those secants. When restricted to a secant of the twisted cubic, the map is conjugate in $PGL(2, F)$ to the map $z \mapsto z^2$. The properties of this map are discussed more thoroughly in section 5, with proofs for geometry over an algebraically closed field and a discussion of the real case.

Pappus' theorem can be seen as a degenerate version of Pascal's theorem. Suppose the triples of points $x = (X_1, X_2, X_3)$ and $y = (Y_1, Y_2, Y_3)$ lie on a conic $\mathcal{C}$ and all points are distinct, then we have:

Theorem 1.5 (Pascal's Theorem)**.** *The points* $Z_1 = \overline{X_2Y_3} \cap \overline{X_3Y_2}$*,* $Z_2 = \overline{X_3Y_1} \cap \overline{X_1Y_3}$*, and* $Z_3 = \overline{X_1Y_2} \cap \overline{X_2Y_1}$ *are colinear.*

We again define $\ell(x, y) = \overline{Z_1Z_2Z_3}$ to be the map which takes two triples of points on a conic to this line. If we define σ and τ of S_3 as before then we can prove an analog of Steiner's Theorem I.

Theorem 1.6 (Steiner's Theorem I for Conics)**.** *The three lines* $\ell(x, y)$*,* $\ell(x, \sigma y)$*, and* $\ell(x, \sigma^2 y)$ *are concurrent and the three lines* $\ell(x, \tau y)$*,* $\ell(x, \tau\sigma y)$*, and* $\ell(x, \tau\sigma^2 y)$ *are concurrent.*

Unfortunately, the further results of Steiner's theorem fail for obvious reasons (there are no lines to return to). A proof of Steiner's theorem for conics similar in spirit to the one we offer can be found in [Ped70].

2. Analytic Projective Geometry

Projective n-space over a field F, $F\mathbf{P}^n$, is the space of all lines through the origin in the vector space F^{n+1}. We use a column vector of homogeneous coordinates, ${}^{\mathrm{T}}(a_0 : \ldots : a_n)$ defined up to scalar multiplication, to describe a point in this space. Lines in these projective spaces are simply all such lines contained in a plane through the origin. The space of projective transformations of $F\mathbf{P}^n$ is $PGL(n+1, F)$, the space of all invertible $n+1$-by-$n+1$ matrices ($GL(n+1, F)$) modulo scalar multiplication. The matrices of $PGL(n+1, F)$ act by left multiplication on the points in $F\mathbf{P}^n$. $PGL(n+1, F)$ acts $n+2$-transitively on points in general position in $F\mathbf{P}^n$, so for example, $PGL(2, F)$ can map any three points in $F\mathbf{P}^1$ to any other three points. Projective geometry gets its name because projections generate these group actions.

We use row vectors of homogeneous coordinates to denote hyperplanes (elements of the dual projective space, $(F\mathbf{P}^n)^*$). A point X lies on a hyperplane Y if and only if the dot product of the corresponding vectors is zero.

$$dot((y_0 : \cdots : y_n), {}^{\mathrm{T}}(x_0 : \ldots : x_n)) = x_0y_0 + \ldots + x_ny_n$$

$PGL(n+1, F)$ acts dually on hyperplanes by right multiplication by inverse matrices, preserving *dot*.

Analytically, a duality D is a pair of maps determined by a matrix $D_M \in PGL(n+1, F)$ for which:

$$D: \begin{array}{c} F\mathbf{P}^n \to (F\mathbf{P}^n)^* : P \mapsto {}^{\mathrm{T}}PD_M \\ (F\mathbf{P}^n)^* \to F\mathbf{P}^n : l \mapsto D_M^{-1}{}^{\mathrm{T}}l \end{array}$$

These maps preserve the incidence properties in the sense that if P is a point in the hyperplane $\mathcal{P}$ then $D(\mathcal{P}) \in D(P)$, since $dot(\mathcal{P}, P) = dot(D(P), D(\mathcal{P}))$ up to scalar multiplication. In addition, it is natural to require that $D \circ D$ is the identity transformation. Indeed, we define a *polarity* as such a duality. In this case, we see that we must have that $D_M = {}^{\mathrm{T}}D_M$. Since we are dealing with equality in $PGL(n+1, F)$, the preimages of D_M in $GL(n+1, F)$ either has the property $M = {}^{\mathrm{T}}M$ or $M = -{}^{\mathrm{T}}M$. The polarity D is known as *symmetric* or *skew-symmetric* respectively depending upon this preimage. The skew-symmetric polarities can only occur in odd dimensional projective spaces, as they come from non-degenerate skew-symmetric matrices. Symmetric polarities are characterized by a classical theorem which we state below. This theorem provides a method of geometrically constructing the symmetric polarities, while our skew hexagon provides a method for constructing the skew-symmetric polarities in $F\mathbf{P}^3$. An important property of skew-symmetric polarities is that they always map a point to a hyperplane containing that point. Up to coordinate changes their are only two polarities in odd dimensions, a skew-symmetric and a symmetric polarity.

Theorem 2.1 (Symmetric Polarities)**.** *Symmetric polarities of $F\mathbf{P}^n$ are in* 1-1 *correspondence with quadric hypersurfaces via the map $D \mapsto \{P \in F\mathbf{P}^n | P \in D(P)\} = \mathcal{Q}_D$. Moreover, for each $P \in \mathcal{Q}_D$, $D(P)$ is the tangent to $\mathcal{Q}_D$ at P. For each $P \in F\mathbf{P}^n$,*

$$D(P) = \bigcup_{\{A,B\in\mathcal{Q}|P\in\overline{AB}\}} D(A) \cap D(B)$$

For an understanding of this result in the plane, look at chapter 8 of [Cox64]. A treatment of the 3-dimensional version is available in section 4.3 of [Todd58].

3. The Linearity Lemma and Pappus' Theorem

Let $\mathbf{M}$ be the space of all non-zero two-by-two matrices modulo scalar multiplication. $\mathbf{M}$ is canonically isomorphic to $F\mathbf{P}^3$ via the map

$$\mathcal{I} : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto {}^{\mathrm{T}}(a : b : c : d) \tag{2}$$

We will use matrices to coordinatize elements of $\mathbf{M}$, but we will think of $\mathbf{M}$ as endowed with the additional structure of $F\mathbf{P}^3$. For example, we will give dual coordinates to the planes of $\mathbf{M}$ (ie. the planes of $F\mathbf{P}^3$ which are canonically identified with some subset of $\mathbf{M}$ via $\mathcal{I}^{-1}$). Define the plane

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \left\{ \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} \in \mathbf{M} \middle| am_{11} + bm_{12} + cm_{21} + dm_{22} = 0 \right\} \in \mathbf{M}^* \tag{3}$$

These coordinates may seem unnatural, but the equation at the right is just the pull back of the dot product, which we call $dot_{\mathbf{M}}$. For $M \in \mathbf{M}$ and $N \in \mathbf{M}^*$, $dot_{\mathbf{M}}(N, M) = dot(\mathcal{I}^*(N), \mathcal{I}(M))$, where $\mathcal{I}^* : \mathbf{M}^* \to (F\mathbf{P}^3)^*$ is defined essentially identically to $\mathcal{I}$ above. For a relevant example, the space of trace zero matrices in $\mathbf{M}$, denoted $\mathsf{Tr_0} \in \mathbf{M}^*$, forms a planar subset of $\mathbf{M}$ and is given the coordinates of the identity matrix:

$$\mathsf{Tr_0} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \left\{ \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} \in \mathbf{M} \middle| m_{11} + m_{22} = 0 \right\} \tag{4}$$

Elements of $PGL(2, F)$ act on $\mathbf{M}$ both on the left and the right, and act dually on the space of planes by multiplication on the same side by the inverse transpose of the element. That is if $M \in \mathbf{M}$ and $M' \in \mathbf{M}^*$ and $\gamma \in PGL(2, F)$ then

$$\begin{array}{ll} (\gamma\cdot) : M \mapsto \gamma \cdot M & (\cdot\gamma) : M \mapsto M \cdot \gamma \\ (\gamma\cdot) : M' \mapsto {}^{\mathrm{T}}\gamma^{-1} \cdot M' & (\cdot\gamma) : M' \mapsto M' \cdot {}^{\mathrm{T}}\gamma^{-1} \end{array}$$

where $(\gamma\cdot)$ and $(\cdot\gamma)$ refer to left and right matrix multiplication respectively. These rules can be derived to be necessary from the facts that $dot_{\mathbf{M}}(N, M) = trace({}^{\mathrm{T}}N \cdot M)$ and the actions must preserve $dot_{\mathbf{M}}$ up to scalar multiplication. These left and right actions are projective and yield embeddings of $PGL(2, F)$ into $PGL(4, F)$ in accordance with the identification $\mathcal{I}$ of $\mathbf{M}$ with $F\mathbf{P}^3$. Also, the map $M \mapsto M^{-1}$ for $M \in PGL(2, F)$ extends to a linear action on matrices preserving $dot_{\mathbf{M}}$, and hence to a projective action on $\mathbf{M}$:

$$^{-1} : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \tag{5}$$

Example 1 (Sample calculation)**.** *Let $V \in \mathbf{M}$ and $H \in \mathbf{M}^*$ be given by*

$$V = \begin{pmatrix} 2 & -1 \\ 1 & 4 \end{pmatrix} \in \mathbf{M} \quad \text{and} \quad H = \begin{pmatrix} -2 & 3 \\ 3 & 1 \end{pmatrix}$$

You can tell that V is a point lying on the plane H because

$$dot_{\mathbf{M}}(H, V) = (-2)(2) + (3)(-1) + (3)(1) + (1)(4) = 0$$

Now suppose $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PGL(2, F)$, then M acts on $\mathbf{M}$ on the left yielding

$$(M\cdot)(V) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 2 & -1 \\ 1 & 4 \end{pmatrix} = \begin{pmatrix} 2a+b & -a+4b \\ 2c+d & -c+4d \end{pmatrix} \in \mathbf{M}$$

by usual matrix multiplication. M acts on $\mathbf{M}^*$ on the left by its inverse transpose

$$(M\cdot)(H) = \begin{pmatrix} d & -c \\ -b & a \end{pmatrix} \begin{pmatrix} -2 & 3 \\ 3 & 1 \end{pmatrix} = \begin{pmatrix} -2d-3c & 3d-c \\ 2b+3a & -3b+a \end{pmatrix} \in \mathbf{M}^*$$

Notice that here we have used a projective inverse of M, rather than its inverse inside $GL(2, F)$ for ease of computation. We will compute that $(M\cdot)(V) \in (M\cdot)(H)$.

$$\begin{aligned} dot_{\mathbf{M}}((M\cdot)(H), (M\cdot)(V)) &= (-2d-3c)(2a+b) + (3d-c)(-a+4b) + \\ &\quad (2b+3a)(2c+d) + (-3b+a)(-c+4d) = 0 \end{aligned}$$

An important subspace of $\mathbf{M}$ is the determinantal quadric $\mathcal{Q}$ consisting of the projectivization of the set of all matrices of determinant zero. Properties of this quadric will be important to the proof of Steiner's theorem, in particular we will need a concrete version of theorem 2.1.

Proposition 3.1 (Determinantal Polarity)**.** *The symmetric polarity $D_{\mathcal{Q}}$ determined by the determinantal quadric $\mathcal{Q}$ sends $M \in \mathbf{M}$ to ${}^{T}M^{-1} \in \mathbf{M}^*$ and sends $N \in \mathbf{M}^*$ to ${}^{T}N^{-1} \in \mathbf{M}$. Here ${}^{-1}$ is a linear rather than group theoretic action as in equation (5).*

Proof. Let $D_{\mathcal{Q}} : \mathbf{M} \to \mathbf{M}^* : M \mapsto {}^{\mathrm{T}}M^{-1}$ and $D_{\mathcal{Q}} : \mathbf{M}^* \to \mathbf{M} : N \mapsto {}^{\mathrm{T}}N^{-1}$. We will show this is a symmetric polarity determined from $\mathcal{Q}$ in the sense of theorem 2.1. If $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{M}$ and $N = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \in \mathbf{M}^*$ then

$$D_{\mathcal{Q}}(M) = \begin{pmatrix} d & -c \\ -b & a \end{pmatrix} \quad \text{and} \quad D_{\mathcal{Q}}(N) = \begin{pmatrix} d' & -c' \\ -b' & a' \end{pmatrix}$$

so $dot_{\mathbf{M}}(N, M) = dot_{\mathbf{M}}(D_{\mathcal{Q}}(M), D_{\mathcal{Q}}(N))$, and so $D_{\mathcal{Q}}$ is a duality. $D_{\mathcal{Q}}^2$ is the identity, so $D_{\mathcal{Q}}$ is a polarity. We compute $\{M \in \mathbf{M} | M \in D_{\mathcal{Q}}(M)\}$ as

$$\{M \in \mathbf{M} | dot_{\mathbf{M}}(D_{\mathcal{Q}}(M), M) = 0\} = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} | 2ad - 2bc = 0\} = \mathcal{Q}$$

Thus $D_{\mathcal{Q}}$ is symmetric because $\{M \in \mathbf{M} | M \in D_{\mathcal{Q}}\} \neq \mathbf{M}$ and by theorem 2.1, $D_{\mathcal{Q}}$ is the symmetric polarity determined by $\mathcal{Q}$. □

Remark 1 ($(\gamma\cdot)$ and $(\cdot\gamma)$ commute with $D_{\mathcal{Q}}$)**.** *We show* $(\gamma\cdot)^{-1} \circ D_{\mathcal{Q}} \circ (\gamma\cdot) = D_{\mathcal{Q}}$. *Consider* $M \in \mathbf{M}$

$$(\gamma\cdot)^{-1} \circ D_{\mathcal{Q}} \circ (\gamma\cdot)(M) = (\gamma^{-1}\cdot)({}^T(\gamma M)^{-1}) = {}^T\gamma({}^T\gamma^{-1}\,{}^TM^{-1}) = {}^TM^{-1} = D_{\mathcal{Q}}(M)$$

because $D_{\mathcal{Q}} \circ (\gamma\cdot)(M) \in \mathbf{M}^*$ *so here the action of* $(\gamma^{-1}\cdot)$ *is by inverse transpose. Similarly* $D_{\mathcal{Q}}$ *commutes with* $(\cdot\gamma)$.

Now we are prepared to give the Linearity Lemma, which will be our fundamental tool in investigating Steiner's theorem. Take lines x_l and y_l as in the start of section 1 containing the triples of points x and y respectively. Now, we construct a map, Π_x^y, from the Zariski open subset of the plane, $F\mathbf{P}^2 \setminus (x_l \cup y_l)$, to $PGL(2, F)$ as follows:

1) Given a point $P \in F\mathbf{P}^2$ construct the linear projection through P of each Y_i onto x_l as $Y_i^P = \overline{Y_i P} \cap x_l$ for $i = 1, 2, 3$.
2) Define $\Pi_x^y(P) \in PGL(2, F)$ to be the unique element for which $\Pi_x^y(P)(X_i) = Y_i^P$ for all i, which exists by the 3-transitivity of $PGL(2, F)$.

For us, a *projective embedding* is an embedding of one projective space into another, that arises from projectivizing a 1-1 linear map between two vector spaces.

Lemma 3.2 (Linearity Lemma)**.** *The map* $\Pi_x^y : F\mathbf{P}^2 \setminus (x_l \cup y_l) \to PGL(2, F)$ *given by* $P \mapsto \Pi_x^y(P)$ *extends to a projective embedding* $F\mathbf{P}^2 \to \mathbf{M} \cong_{\mathcal{I}} F\mathbf{P}^3$

We will break up the proof of this important lemma into a pair of propositions:

Proposition 3.3. *The map* $\Pi_x^y : F\mathbf{P}^2 \setminus (x_l \cup y_l) \to PGL(2, F)$ *maps (Zariski open subsets of) lines to (Zariski open subsets of) lines. The image of* Π_x^y *is contained in a plane.*

Proof. Given points $S, T \in x_l$, consider the set of all $\pi \in PGL(2, F)$ for which $\pi(S) = T$. In terms of matrices and up to scalar multiplication this means:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} S_1 \\ S_2 \end{pmatrix} = \begin{pmatrix} T_1 \\ T_2 \end{pmatrix} \implies T_2(aS_1 + bS_2) = T_1(cS_1 + dS_2)$$

Clearly, this yields a linear relationship among the matrix elements and this relationship is determined uniquely by S and T. We will use this linear relationship without referring to an explicit equation.

Now choose a line l not through $O = x_l \cap y_l$ and $P \in l$. Let $\alpha \in PGL(3, F)$ be a projective transformation of the plane which takes the points X_i to Y_i (such a projective transformation exists and has a unique action when restricted to x_l by the 3-transitivity of $PGL(2, F)$). Let π_P be the map which takes a point on y_l to the point on x_l obtained by projection through P. We see that $\Pi_x^y(P) = \pi_P \circ \alpha$.

Clearly $\pi_P(O) = O$ and since $P \in l$, $\pi_P(l \cap y_l) = l \cap x_l$. Thus for all P on l, $\Pi_x^y(P) \in PGL(2,F)$ satisfies the two conditions:

$$\Pi_x^y(P)(\alpha^{-1}O) = O \tag{6}$$

$$\Pi_x^y(P)(\alpha^{-1}(l \cap y_l)) = l \cap x_l \tag{7}$$

Each of these equations determines a distinct planar subset of $\mathbf{M} \cong F\mathbf{P}^3$ by the discussion of the previous paragraph, and therefore these two planes intersect in a line. Condition (6) above actually holds independently of the choice of l, thus we see the image of Π_x^y lies in the plane determined by condition (6).

At this point we could probably use various machinery from algebraic geometry on Zariski open sets together with the fundamental theorem of projective geometry to prove our lemma, but this would veer away from the goal to provide an elementary proof.

We still must prove that Π_x^y maps a line l through O to a line. Choose $P, Q \in l$. As before $\Pi_x^y(P) = \pi_P \circ \alpha$ and $\Pi_x^y(Q) = \pi_Q \circ \alpha$. Consider $\Pi_x^y(P) \circ \Pi_x^y(Q)^{-1} = \pi_P \circ \pi_Q^{-1}$ corresponding geometrically to projection from x_l through Q to y_l followed by projection through P back to x_l. We claim that this operation has a single fixed point, namely O. Suppose $X \in x_l \smallsetminus \{O\}$ is fixed. Then set $\pi_Q^{-1}(X) = Y \in y_l \smallsetminus \{O\}$, so in particular $X \neq Y$. $\pi_Q^{-1}(X) = Y$ implies X, Y, and Q are colinear and $\pi_P(Y) = X$ implies X, Y, and P are colinear. Finally since $X \neq Y$, all four points must be colinear, but this is absurd since both P and Q lie on l and $\overline{XY}$ so $P = Q = l \cap \overline{XY}$.

We have shown that the matrix corresponding to $\Pi_x^y(P) \circ \Pi_x^y(Q)^{-1}$ has a single eigenvector for any $P \in \overline{OQ} = l$, namely O. Fix a $C \in PGL(2,F)$ for which $C(O) = {}^{\mathrm{T}}(1:0)$. We see $\Pi_x^y(P) \circ \Pi_x^y(Q)^{-1}$ is conjugate via C to (a projective analog of) its Jordan canonical form:

$$\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} = C \circ \Pi_x^y(P) \circ \Pi_x^y(Q)^{-1} \circ C^{-1}$$

the set of all which forms a line in $\mathbf{M}$. Solving for $\Pi_x^y(P)$, we see that for all $P \in l$, there is an $a, b \in F$ such that

$$\Pi_x^y(P) = C^{-1} \circ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \circ C \circ \Pi_x^y(Q) \tag{8}$$

Matrix multiplication on the right and left acts linearly on $\mathbf{M}$, so $\Pi_x^y(l)$ is a line in $\mathbf{M}$. □

The next proposition will describe the completion of the map $\Pi_x^y : F\mathbf{P}^2 \setminus (x_l \cup y_l) \to PGL(2,F)$ to a map from $F\mathbf{P}^2 \to \mathbf{M}$. The image of the remaining points of $F\mathbf{P}^2$ will be contained in $\mathbf{M} \setminus PGL(2,F) = \mathcal{Q}$, that is the space of non-zero matrices with zero determinant modulo scalar multiplication. Before we state this proposition, let us review briefly some properties of these matrices. Such a matrix can be generically written as

$$M = \begin{pmatrix} ac & bc \\ ad & bd \end{pmatrix} \tag{9}$$

Further such a matrix's equivalence class is determined by its kernel and image in $F\mathbf{P}^1$. Namely if

$$\overrightarrow{v} = \begin{pmatrix} -b \\ a \end{pmatrix} \quad \overrightarrow{w} = \begin{pmatrix} c \\ d \end{pmatrix}$$

then $M(\overrightarrow{v}) = 0$ and for all $\overrightarrow{u}$ not a scalar multiple of $\overrightarrow{v}$ we have that $M(\overrightarrow{u})$ is a nonzero scalar multiple of $\overrightarrow{w}$. We have a bijective map to the determinantal quadric $\mathcal{Q} = \mathbf{M} \setminus PGL(2, F)$:

$$\begin{array}{cccc} Q: & F\mathbf{P}^1 \times F\mathbf{P}^1 & \rightarrow & \mathcal{Q} \\ Q: & (v, w) & \mapsto & M \end{array} \tag{10}$$

where v and w are the points in $F\mathbf{P}^1$ corresponding to the scalar equivalence class of $\overrightarrow{v}$ and $\overrightarrow{w}$ and M is as above in equation (9). The map Q is linear in each coordinate, therefore the $Q(v, F\mathbf{P}^1)$ and $Q(F\mathbf{P}^1, w)$ are lines in the determinantal quadric. We will utilize this map in the statement of the following proposition:

Proposition 3.4. $\Pi_x^y : F\mathbf{P}^2 \setminus (x_l \cup y_l) \rightarrow PGL(2, F)$ *naturally extends to a line preserving map* $F\mathbf{P}^2 \rightarrow \mathbf{M}$ *by defining* $\Pi_x^y(X) = Q(\alpha^{-1}(O), X)$ *for* $X \in x_l$ *and* $\Pi_x^y(X) = Q(\alpha^{-1}(Y), O)$ *for* $Y \in y_l$. *Here* $O = x_l \cap y_l$ *and* $\alpha \in PGL(3, F)$ *is a projective transformation of the plane sending the triple* x *to the triple* y *as in the proof above.*

Proof. Suppose l is a line passing through x_l and y_l at distinct points $X = x_l \cap l$ and $Y = y_l \cap l$ respectively. Recall equations (6) and (7) yield distinct linear conditions determining the image of l. We will show X and Y also satisfy these conditions. The first condition was for $P \in l$, $\Pi_x^y(P)(\alpha^{-1}O) = O$. From our definition of $\Pi_x^y(X)$, its kernel is $\alpha^{-1}O$, so $\Pi_x^y(X)(\alpha^{-1}O) = 0$ and this first linear condition is satisfied degenerately. We defined $\Pi_x^y(Y)$ so that its image is $\{O\}$, thus in particular the equation $\Pi_x^y(Y)(\alpha^{-1}O) = O$ is satisfied. The second condition (equation (7)) can be rewritten as $\Pi_x^y(P)(\alpha^{-1}(Y)) = X$. Since $\Pi_x^y(X)$ was defined so that its image is X, this condition is clearly satisfied for X. $\Pi_x^y(Y)$ has a kernel consisting of $\alpha^{-1}(Y)$, so $\Pi_x^y(Y)(\alpha^{-1}(Y)) = 0$ and the second condition is satisfied degenerately.

We have shown that our extension of Π_x^y sends lines which pass through x_l and y_l at distinct points to lines. It is now necessary to show that this extension sends a line l through O to a line. Choose Q on l and the conjugacy matrix C as in the proof of proposition 3.3. We need to show that O is the unique eigenvector for $\Pi_x^y(O) \circ \Pi_x^y(Q)^{-1}$. As before we can write $\Pi_x^y(Q) = \pi_Q \circ \alpha$ where π_Q is projection from y_l to x_l through Q. Then $\Pi_x^y(Q)^{-1} = \alpha^{-1} \circ \pi_Q^{-1}$ and

$$\Pi_x^y(Q)^{-1}(O) = \alpha^{-1} \circ \pi_Q^{-1}(O) = \alpha^{-1}(O)$$

Since we defined $\Pi_x^y(O) = Q(\alpha^{-1}(O), O)$, it follows that $\Pi_x^y(O) \circ \Pi_x^y(Q)^{-1}(O) = 0$ while for all points $X \in x_l$ with $X \neq O$, $\Pi_x^y(O) \circ \Pi_x^y(Q)^{-1}(X) = O$. Thus O is the unique eigenvector for $\Pi_x^y(O) \circ \Pi_x^y(Q)^{-1}$, with eigenvalue zero. Substituting O for P in equation (8) we would see the equation is satisfied when a=0. Therefore our definition for $\Pi_x^y(O)$ makes it colinear with the rest of $\Pi_x^y(l)$. □

Now we may conclude the proof of the Linearity lemma:

Proof of Lemma 3.2. The map $\Pi_x^y : F\mathbf{P}^2 \rightarrow \mathbf{M}$ is an algebraic 1-1 map, mapping a plane to a plane and mapping lines to lines. Therefore by the fundamental theorem of projective geometry, Π_x^y is a projective embedding, that is Π_x^y is the projectivization of a linear map. ☐

There is also a linearity lemma for conics. The subgroup of $PGL(3,F)$ preserving a conic is isomorphic to $PGL(2,F)$, and $PGL(2,F)$ also acts 3-transitively on points on conics. If $x = (X_1, X_2, X_3)$ and $y = (Y_1, Y_2, Y_3)$ are triples of points on a conic $\mathcal{C} \subset F\mathbf{P}^2$, we redefine Π_x^y accordingly:

1) Given a point $P \in F\mathbf{P}^2 \setminus \mathcal{C}$ construct the points $Y_1^P\ Y_2^P\ Y_3^P$ on $\mathcal{C}$ as $Y_i^P = (\overline{Y_i P} \smallsetminus \{Y_i\}) \cap \mathcal{C}$.
2) Define $\Pi_x^y(P) \in PGL(2,F)$ to be the unique element for which $\Pi_x^y(P)(X_i) = Y_i^P$ for $i = 1, 2, 3$.

Lemma 3.5 (Linearity Lemma for Conics). *The map $\Pi_x^y : F\mathbf{P}^2 \setminus \mathcal{C} \to PGL(2,F)$ given by $P \mapsto \Pi_x^y(P)$ extends to a projective embedding $F\mathbf{P}^2 \to \mathbf{M}$.*

Rather than prove this lemma, we will just make some comments on the proof.

Sketch of proof. Define $\alpha \in F\mathbf{P}^2$ to be the unique projective transformation mapping x to y. Then for all P, $\Pi_x^y(P) = \pi_P \circ \alpha$ where π_P is the involution obtained by projection through P, which sends a point $X \in \mathcal{C}$ to $\overline{PX} \cap \mathcal{C} \smallsetminus \{X\}$. It can be easily verified that such a map extends to a projective transformation of the plane preserving $\mathcal{C}$ and thus gives rise to an element of $PGL(2,F)$ of order 2. Thus for all P, $\Pi_x^y(P) \circ \alpha^{-1}$ is an involution. But all involutions in $PGL(2,F)$ have trace zero, thus $(\cdot\alpha^{-1})\Pi_x^y(F\mathbf{P}^2) \subset \mathsf{Tr}_0 = I$ by equation (4) so that $\Pi_x^y(F\mathbf{P}^2) \subset {}^{\mathrm{T}}\alpha^{-1}$.

Suppose a line $l \subset F\mathbf{P}^2$ intersects $\mathcal{C}$ at two points, X and Y. Then for all P on l, π_P exchanges X and Y. This gives rise to two linear equation determining that the image of l under Π_x^y is contained a line in $\mathbf{M}$. If l intersects $\mathcal{C}$ at a single point X, then for all P on l, π_P preserves X. This together with the fact that π_P is trace zero determines the linear image of l.

We extend Π_x^y to $\mathcal{C}$ by defining $\Pi_x^y(P) = Q(P,P) \circ \alpha$ for $P \in \mathcal{C}$, where Q is the map as defined above proposition 3.4. This extension remains line preserving and thus by the fundamental theorem of projective geometry is a projective embedding. ☐

Remark 2. *$PGL(2,F)$ embeds as a Zariski open set into $\mathbf{M}$, as the complement of the determinantal variety ($ad - bc = 0$), a quadric surface in $\mathbf{M}$. By the linearity lemma, Π_x^y embeds the projective plane into $\mathbf{M}$ as a planar subset. For the conic case, $\Pi_x^y(P) \in PGL(2,F)$ when $P \notin \mathcal{C}$, thus $\Pi_x^y(F\mathbf{P}^2)$ intersects the determinantal variety in a conic. When x and y lie on two lines, then $\Pi_x^y(F\mathbf{P}^2)$ must intersect the determinantal variety in two lines, and thus is tangent to this quadric.*

As a demonstration of the power of the Linearity Lemmas, we will now prove Pappus' and Pascal's theorems. First, the involutions (elements of order 2) of $PGL(2,F)$ have properties worth mentioning. Any element of $PGL(2,F)$ whose action swaps two points is an involution (this is exercise 6.7 in [KK96]). Here is a cheap proof. Suppose $\gamma \in PGL(2,F)$ swaps A and B. We know every projective transformation has a fixed point (inside an algebraically closed field), therefore γ^2 fixes this fixed point, A, and B, so by the 3-transitivity of $PGL(2,F)$, γ^2 is the identity. We will also need the fact that an element of $PGL(2,F)$ is an involution if and only if it has trace zero. This follows from manipulating equation (5). Now as a corollary of our work, we have a proof of theorems of Pappus and Pascal (theorems 1.1 and 1.5):

Proof of Pappus' and Pascal's theorems. The general attack will be to show that $\Pi_x^y(Z_i)$ is an involution for all i, hence each $\Pi_x^y(Z_i)$ is contained in the plane Tr_0 consisting of all points in $\mathbf{M}$ with trace zero. Then $\Pi_x^y(Z_i) \in \mathsf{Tr}_0 \cap \Pi_x^y(F\mathbf{P}^2)$ which is a line by the linearity lemma, and so each Z_i is contained in the pullback of this line via $(\Pi_x^y)^{-1}$ which by another application of the linearity lemma shows us the points Z_1, Z_2, and Z_3 are colinear.

By the paragraph above, it is sufficient to show that each $\Pi_x^y(Z_i)$ is an involution, so let us consider the actions $\Pi_x^y(Z_i)$. I claim that these maps act as involutions. This can be seen fairly easily. We defined $Z_1 = \overline{X_2Y_3} \cap \overline{X_3Y_2}$. Now, consider $\Pi_x^y(Z_1)(X_2)$. This is defined to be $\overline{Y_2Z_1} \cap x_l$, which is X_3 (see figure 1). Similarly, $\Pi_x^y(Z_1)(X_3) = X_2$. Thus $\Pi_x^y(Z_1)$ swaps the points X_2 and X_3 so is an involution. Identical arguments show that $\Pi_x^y(Z_2)$ and $\Pi_x^y(Z_3)$ are involutions.

Our only fear is that perhaps $\mathsf{Tr}_0 = \Pi_x^y(F\mathbf{P}^2)$. To discount this, first notice that Tr_0 intersects the $\mathcal{Q}$ in a conic, thus this could only happen in the conic case, by remark 2. Now recalling that $\Pi_x^y(F\mathbf{P}^2) = {}^{\mathrm{T}}\alpha^{-1}$ with $\alpha \in PGL(2,F)$ mapping $x \mapsto y$ projectively, we see that equation (4) and $\Pi_x^y(F\mathbf{P}^2) = \mathsf{Tr}_0$ implies $\alpha = I$ and so x must equal y. But $x = y$ is absurd for our construction. $\square$

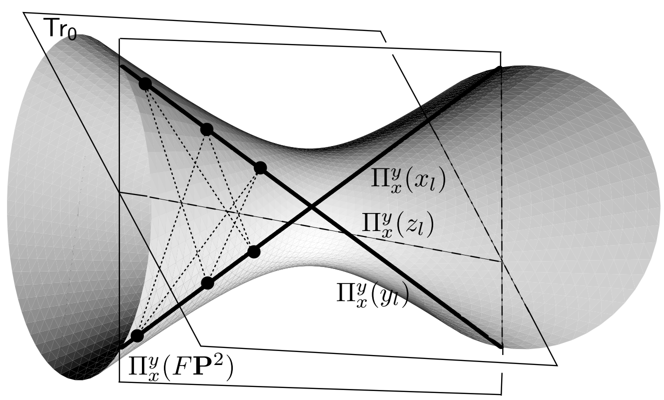

FIGURE 2. Proof of Pappus' Theorem. The quadric represents $\mathbf{M} \setminus PGL(2,F)$.

Corollary 3.6 (Pappus' and Pascal's Line)**.** *A point P lies on the line $\overline{Z_1Z_2Z_3}$ constructed by Pappus' Theorem or Pascal's Theorem if and only if $\Pi_x^y(P)$ has trace zero (or equivalently is an involution).*

Remark 3. *The linearity lemma generalizes (but not the version regarding conics) to the case of two hyperplanes in $F\mathbf{P}^n$ with $n+1$-tuples of points on them in general position. The higher dimensional generalization of Pappus' theorem in* [Wit79] *can be proven by considering trace zero matrices in $PGL(n,F)$. In fact, an analogous proof to the proof we will give of Steiner's Theorem I will also go through in this case, though its further claims fail.*

4. Steiner's Theorem

In this section, we will provide proofs of the theorems over algebraically closed fields. This entails the results in general, because any field can be embedded in its algebraic closure. A statement only involving linear subspaces which is true in projective space over a field is also true over any subfield.

To begin our investigation of Steiner's theorem, we are interested in how permutations of the triple y affect the map Π_x^y. Choose a generic point P. We defined $Y_i^P = \overline{Y_i P} \cap x_l$. Thus for $\upsilon \in S_3$, $(\upsilon Y)_i^P = \overline{Y_{\upsilon^{-1}(i)} P} \cap x_l$. Consequently, $\Pi_x^{\upsilon y}(P)(X_i) = \overline{Y_{\upsilon^{-1}(i)} P} \cap x_l$.

Given x we get a representation ρ_x of S_3 into $PGL(2,F)$ defined such that

$$\rho_x(\upsilon)(x) = \upsilon^{-1} x \tag{11}$$

or equivalently such that $\rho_x(\upsilon)(X_i) = X_{\upsilon(i)}$ for all i. This is well defined because $PGL(2,F)$ acts 3-transitively on x_l. Then we have that

$$\Pi_x^y(P) \circ \rho_x(\upsilon^{-1})(X_i) = \Pi_x^y(P)(X_{\upsilon^{-1}(i)}) = \overline{Y_{\upsilon^{-1}(i)} P} \cap x_l = \Pi_x^{\upsilon(y)}(P)(X_i)$$

Thus we have shown $\Pi_x^y(P) \cdot \rho_x(\upsilon^{-1}) = \Pi_x^{\upsilon(y)}(P)$. It can be verified that this is even true when $\Pi_x^y(P)$ or $\Pi_x^{\upsilon(y)}(P)$ are elements of $\mathbf{M} \smallsetminus PGL(2,F)$.

Recall that $\ell(x,y)$ has the property that $\Pi_x^y(\ell(x,y)) \subset \mathsf{Tr}_0$ by corollary 3.6 above. This means that $\Pi_x^{\upsilon y}(\ell(x,\upsilon y)) \subset \mathsf{Tr}_0$ and therefore $\Pi_x^y(\ell(x,\upsilon y)) \cdot \rho_x(\upsilon^{-1}) \subset \mathsf{Tr}_0$. $PGL(2,F)$ acts on $\mathbf{M}^*$ by inverse transpose, so because $\mathsf{Tr}_0 = I$ as in equation (4)

$$\Pi_x^y(\ell(x,\upsilon(y))) \subset {}^{\mathrm{T}}\rho_x(\upsilon^{-1}) \in \mathbf{M}^*$$

To simplify notation we define the planes

$$H_\upsilon = {}^{\mathrm{T}}\rho_x(\upsilon^{-1}) \in \mathbf{M}^* \forall \upsilon \in S_3 \tag{12}$$

We are now prepared to offer proof of theorems 1.2 and 1.6:

Proof of Steiner's Theorem I. We wish to show that $\ell(x,y)$, $\ell(x,\sigma y)$ and $\ell(x,\sigma^2 y)$ intersect at a common point for $\sigma = (123) \in S_3$. We have shown that $\Pi_x^y(\ell(x,y)) \subset H_e = \mathsf{Tr}_0$, $\Pi_x^y(\ell(x,\sigma y)) \subset H_\sigma$, and $\Pi_x^y(\ell(x,\sigma^2 y)) \subset H_{\sigma^2}$. We will prove that the three planes H_e H_σ and H_{σ^2} intersect in a line, l_{even}. It will follow that the three lines $\ell(x,y)$, $\ell(x,\sigma y)$ and $\ell(x,\sigma^2 y)$ intersect in a common point namely $(\Pi_x^y)^{-1}(l_{even} \cap \Pi_x^y(F\mathbf{P}^2))$, the pullback of the single point intersection of a line and a plane.

We will now show that if $m \in H_e$ and $m \in H_\sigma$ then $m \in H_{\sigma^2}$. Suppose $m \in \mathbf{M}$ is contained in $H_e = \mathsf{Tr}_0$, then we know m is an involution and thus

$$m = m^{-1} \tag{13}$$

Now suppose $m \in H_\sigma = {}^{\mathrm{T}}\rho_x(\sigma^{-1})$. This is the same as saying $\rho_x(\sigma^{-1}) \cdot m \in \mathsf{Tr}_0$, by acting on both sides by $\rho_x(\sigma^{-1})$ on the left (recall again this acts on the plane H_σ by inverse transpose). Then $\rho_x(\sigma^{-1}) \cdot m$ is an involution, so

$$\rho_x(\sigma^{-1}) \cdot m = (\rho_x(\sigma^{-1}) \cdot m)^{-1} = m^{-1} \cdot \rho_x(\sigma) \tag{14}$$

Substituting equation (13) into equation (14), we see:

$$\begin{aligned} \rho_x(\sigma^{-1}) \cdot m^{-1} &= m \cdot \rho_x(\sigma) \\ (m \cdot \rho_x(\sigma))^{-1} &= m \cdot \rho_x(\sigma) \end{aligned} \tag{15}$$

Which is the same as saying $m \cdot \rho_x(\sigma) \in \mathsf{Tr}_0$ and therefore m is contained in $\mathsf{Tr}_0 \cdot {}^{\mathrm{T}}\rho_x(\sigma) = H_{\sigma^2}$. It follows that $H_e \cap H_\sigma \cap H_{\sigma^2} = H_e \cap H_\sigma$, a line.

At first glance it would appear that this proof depends on $m \in PGL(2,F) \subset \mathbf{M}$, but the equations are always true provided you define $^{-1}$ as in equation (5).

Thus $\ell(x,y)$, $\ell(x,\sigma y)$ and $\ell(x,\sigma^2 y)$ are concurrent. Also, it follows $\ell(x,\tau y)$, $\ell(x,\sigma\tau y)$, and $\ell(x,\sigma^2\tau y)$ are concurrent by substitution of τy for y. □

Following Steiner's theorem, we have constructed the six lines $\ell(x,vy)$, which we will denote by ℓ_v. Via our projective embedding into $\mathbf{M}$, these lines correspond to

$$\Pi_x^y(\ell_v) = \Pi_x^y(F\mathbf{P}^2) \cap H_v$$

We arranged these lines in coincident triples $z_{even} = (\ell_e, \ell_\sigma, \ell_{\sigma^2})$ and $z_{odd} = (\ell_\tau, \ell_{\tau\sigma}, \ell_{\tau\sigma^2})$. Now we wish to consider $\ell^*(z_{even}, z_{odd})$. Explicitly

$$\ell^*(z_{even}, z_{odd}) = \begin{array}{c} \overline{(\ell_\sigma \cap \ell_{\tau\sigma^2})(\ell_{\sigma^2} \cap \ell_{\tau\sigma})} \cap \\ \overline{(\ell_{\sigma^2} \cap \ell_\tau)(\ell_e \cap \ell_{\tau\sigma^2})} \cap \\ \overline{(\ell_e \cap \ell_{\tau\sigma})(\ell_\sigma \cap \ell_\tau)} \end{array}$$

We can reinterpret this construction as embedded inside $\mathbf{M}$ as

$$\Pi_x^y(\ell^*(z_{even}, z_{odd})) = \begin{array}{c} \overline{(H_\sigma \cap H_{\tau\sigma^2} \cap \Pi_x^y(F\mathbf{P}^2))(H_{\sigma^2} \cap H_{\tau\sigma} \cap \Pi_x^y(F\mathbf{P}^2))} \cap \\ \overline{(H_{\sigma^2} \cap H_\tau \cap \Pi_x^y(F\mathbf{P}^2))(H_e \cap H_{\tau\sigma^2} \cap \Pi_x^y(F\mathbf{P}^2))} \cap \\ \overline{(H_e \cap H_{\tau\sigma} \cap \Pi_x^y(F\mathbf{P}^2))(H_\sigma \cap H_\tau \cap \Pi_x^y(F\mathbf{P}^2))} \end{array} \tag{16}$$

We will show that the six lines implicitly shown in equation (16)

$$\begin{array}{cc} H_\sigma \cap H_{\tau\sigma^2} & H_{\sigma^2} \cap H_{\tau\sigma} \\ H_{\sigma^2} \cap H_\tau & H_e \cap H_{\tau\sigma^2} \\ H_e \cap H_{\tau\sigma} & H_\sigma \cap H_\tau \end{array} \tag{17}$$

arrange themselves in a hexagon. The group $S_3 = <\sigma, \tau | \sigma^3 = \tau^2 = (\sigma\tau)^2 = e>$ is generated by two of its involutions, say $\tau\sigma$ and $\tau\sigma^2$. Consider the Cayley graph constructed using these two generators. This Cayley graph has the group elements of S_3 as its vertices, and connects each element $v \in S_3$ to $\tau\sigma v$ and $\tau\sigma^2 v$. This Cayley graph is the hexagon shown in figure 3.

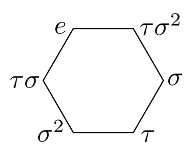

FIGURE 3. The Cayley graph of S_3 with generators $\tau\sigma$ and $\tau\sigma^2$

Remark 4. *There are* 6 *possible hexagonal Cayley graphs of this type for* S_3 *corresponding to the* 6 *points that are constructed for Steiner's theorem II. First we choose the two permutations and second we must choose whether the generators act on the left or the right.*

We will call the hexagon $V_1V_2V_3V_4V_5V_6 \subset F\mathbf{P}^2$ a *skew hexagon* if

- V_1, V_3, and V_5 are contained in a line l,
- V_2, V_4, and V_6 are contained in a line l',
- and l and l' are skew lines.

There is an incarnation of the hexagonal Cayley graph just described and depicted in figure 3 as a skew hexagon composed of the lines in (17) above. Namely for each edge of the Cayley graph, say $\overline{\upsilon\upsilon'}$, construct the line $H_\upsilon \cap H_{\upsilon'}$ (these are the lines of equation (17)). At first glance we have six lines floating in space, but in fact they intersect as suggested by the Cayley graph. This is because two adjacent lines are given by $H_\upsilon \cap H_{\tau\sigma\upsilon}$ and $H_\upsilon \cap H_{\tau\sigma^2\upsilon}$ which intersect at the vertex

$$V_\upsilon = H_\upsilon \cap H_{\tau\sigma\upsilon} \cap H_{\tau\sigma^2\upsilon}$$

This definition forces that for $\upsilon, \upsilon' \in S_3$ adjacent in the Cayley graph we have $H_\upsilon \cap H_{\upsilon'} = \overline{V_\upsilon V_{\upsilon'}}$. The three planes H_υ with υ even intersect in a common line, l_{even}, and the three planes with υ odd intersect in the line l_{odd}. Then $V_\upsilon \in l_{odd}$ for υ even, since $V_\upsilon \in H_{\tau\sigma\upsilon} \cap H_{\tau\sigma^2\upsilon} = l_{odd}$. Similarly, the three points V_υ for υ odd are colinear and contained in the line l_{even}. Our skew hexagon is therefore self dual in the sense that both the hexagon composed of V_υ and H_υ are both skew, just in dual projective spaces. But, our skew hexagon is self dual in a much stronger sense. Recall the definition of skew-symmetric polarity appearing in section 2.

Proposition 4.1 (Skew-symmetry). *There is a skew-symmetric polarity D which acts on our skew hexagon as:*

$$D : H_\upsilon \mapsto V_{\tau\upsilon} \quad D : V_\upsilon \mapsto H_{\tau\upsilon}$$

Proof. We will prove this proposition by brute force. Since $PGL(2, F)$ acts 3-transitively on points on the line x_l, we may choose coordinates for the three points in x arbitrarily, for ease of computation, we choose

$$x = ({}^{\mathrm{T}}(1:1), {}^{\mathrm{T}}(\omega:\omega^2), {}^{\mathrm{T}}(\omega^2:\omega)) \in (F\mathbf{P}^1)^3$$

with $\omega \in F$ a cube root of unity, that is $\omega^3 = 1$ but $\omega \neq 1$. We chose $\sigma = (123)$ and $\tau = (23)$, which give us our representation

$$\rho_x(\sigma) = \begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix} \quad \rho_x(\tau) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

From our definition of $H_\upsilon = {}^{\mathrm{T}}\rho_x(\upsilon^{-1})$ we obtain coordinates for the planes H_υ:

$$H_e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad H_\sigma = \begin{pmatrix} \omega^2 & 0 \\ 0 & \omega \end{pmatrix} \quad H_{\sigma^2} = \begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix}$$

$$H_\tau = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad H_{\tau\sigma} = \begin{pmatrix} 0 & \omega \\ \omega^2 & 0 \end{pmatrix} \quad H_{\tau\sigma^2} = \begin{pmatrix} 0 & \omega^2 \\ \omega & 0 \end{pmatrix}$$

From our definition of $V_\upsilon = H_\upsilon \cap H_{\tau\sigma\upsilon} \cap H_{\tau\sigma^2\upsilon}$ we obtain coordinates for the points V_υ, which can be checked by computing cross products as in equation (3).

$$V_e = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad V_\sigma = \begin{pmatrix} \omega & 0 \\ 0 & -\omega^2 \end{pmatrix} \quad V_{\sigma^2} = \begin{pmatrix} \omega^2 & 0 \\ 0 & -\omega \end{pmatrix}$$

$$V_\tau = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad V_{\tau\sigma} = \begin{pmatrix} 0 & -\omega^2 \\ \omega & 0 \end{pmatrix} \quad V_{\tau\sigma^2} = \begin{pmatrix} 0 & -\omega \\ \omega^2 & 0 \end{pmatrix}$$

Now, we can explicitly state the map D:

$$D : \mathbf{M} \to \mathbf{M}^* : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} b & -a \\ d & -c \end{pmatrix}$$

$$D : \mathbf{M}^* \to \mathbf{M} : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} b & -a \\ d & -c \end{pmatrix}$$

By inspection $D : H_v \mapsto V_{\tau v}$ and $D : V_v \mapsto H_{\tau v}$ by inspection. To see that D is a skew-symmetric polarity, note that D is linear, D preserves the dot product $\mathbf{M} \times \mathbf{M}^* \to F$ up to scalar multiple, D^2 is the identity, and finally that D maps a point in $\mathbf{M}$ to a plane containing that point. This final property differentiates a skew-symmetric polarity from a symmetric polarity and can be checked by computing a dot product:

$$dot_{\mathbf{M}}(D(M), M) = dot_{\mathbf{M}}(\begin{pmatrix} b & -a \\ d & -c \end{pmatrix}, \begin{pmatrix} a & b \\ c & d \end{pmatrix}) = ba - ab + dc - cd = 0$$

□

Before introducing the skew-hexagon theorem, we will need the following geometric property of skew-symmetric polarities restricting how lines are sent to to lines.

Proposition 4.2. *If D is a skew-symmetric polarity swapping skew (non-intersecting) lines l_1 and l_2, then any line m passing through both is preserved by D.*

Proof. Define the planes $\mathcal{P}_i = span(l_i, m)$ and the points $P = l_i \cap m$ for $i = 1, 2$. Then $m = \mathcal{P}_1 \cap \mathcal{P}_2 = \overline{P_1P_2}$ and $l_2 \cap \mathcal{P}_1 = P_2$ and $l_1 \cap \mathcal{P}_2 = P_1$.

Because $l_1 \subset \mathcal{P}_1$ and D is a polarity, we have $D(\mathcal{P}_1) \subset D(l_1) = l_2$. And because D is a skew-symmetric polarity, $D(\mathcal{P}_1) \subset \mathcal{P}_1$. Thus $D(\mathcal{P}_1) = l_2 \cap \mathcal{P}_1 = P_2$. Similarly it follows $D(\mathcal{P}_2) = P_1$. Finally we see m is fixed:

$$D(m) = D(\mathcal{P}_1 \cap \mathcal{P}_2) = \overline{D(\mathcal{P}_1)D(\mathcal{P}_2)} = \overline{P_2P_1} = m$$

□

The skew hexagon theorem will demonstrate how the skew-symmetric polarity D relates to our construction. In itself this theorem represents a concrete construction of skew-symmetric polarities in $F\mathbf{P}^3$ using a skew hexagon. The main observation here is that any skew hexagon is projectively equivalent to any other skew hexagon, therefore every skew hexagon has an associated skew-symmetric polarity just like our hexagon.

Theorem 4.3 (The Skew Hexagon Theorem)**.** *Consider a skew hexagon in $F\mathbf{P}^3$ with vertices $V_1V_2V_3V_4V_5V_6$ such that the odd vertices are colinear and the even vertices are colinear. Any plane $\mathcal{P}$ transverse to each of the edges of the hexagon intersects these edges in vertices of a pair of Desarguesian triangles, that is the triangle A with vertices*

$$A_1 = \overline{V_1V_2} \cap \mathcal{P} \quad A_2 = \overline{V_3V_4} \cap \mathcal{P} \quad A_3 = \overline{V_5V_6} \cap \mathcal{P}$$

and the triangle B with vertices

$$B_1 = \overline{V_4V_5} \cap \mathcal{P} \quad B_2 = \overline{V_6V_1} \cap \mathcal{P} \quad B_3 = \overline{V_2V_3} \cap \mathcal{P}$$

meet the requirements of Desargue's theorem so that $\overline{A_1B_1}$, $\overline{A_2B_2}$, and $\overline{A_3B_3}$ are coincident. Dually, given a general point P in space, the triangle of planes $\mathcal{A}$ composed of

$$\mathcal{A}_1 = span(\overline{V_1V_2}P) \quad \mathcal{A}_2 = span(\overline{V_3V_4}P) \quad \mathcal{A}_3 = span(\overline{V_5V_6}P)$$

and the triangle of planes $\mathcal{B}$ composed of

$$\mathcal{B}_1 = span(\overline{V_4V_5}P) \quad \mathcal{B}_2 = span(\overline{V_6V_1}P) \quad \mathcal{B}_3 = span(\overline{V_2V_3}P)$$

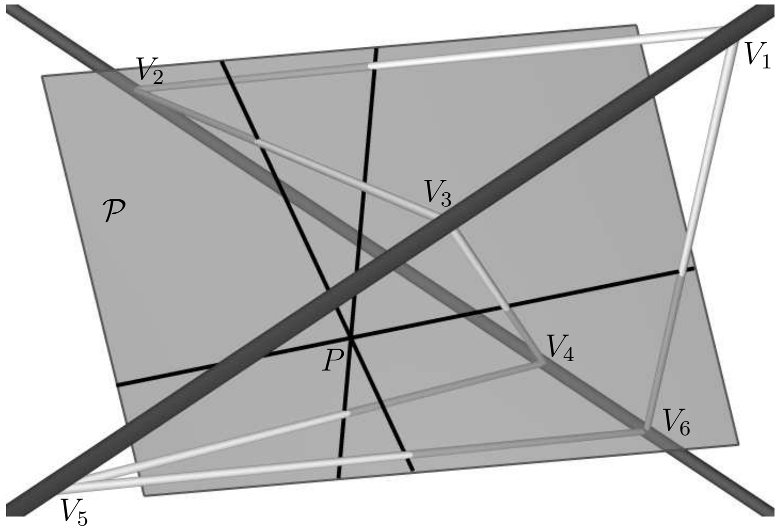

FIGURE 4. Diagram depicting the Skew Hexagon Theorem in $F\mathbf{P}^3$

are Desarguesian triangles in the dual projective plane of the set of all planes through the point P*, that is* $\mathcal{A}_1 \cap \mathcal{B}_1$*,* $\mathcal{A}_2 \cap \mathcal{B}_2$*, and* $\mathcal{A}_3 \cap \mathcal{B}_3$ *are coplanar. The map*

$$\mathcal{P} \mapsto \overline{A_1B_1} \cap \overline{A_2B_2} \cap \overline{A_3B_3} \quad P \mapsto span((\mathcal{A}_1 \cap \mathcal{B}_1)(\mathcal{A}_2 \cap \mathcal{B}_2)(\mathcal{A}_3 \cap \mathcal{B}_3))$$

extends to a skew-symmetric polarity of $F\mathbf{P}^3$*.*

Proof. The hexagon $V_1V_2V_3V_4V_5V_6$ is projectively equivalent to $V_eV_{\tau\sigma}V_{\sigma^2}V_\tau V_\sigma V_{\tau\sigma^2}$ so by proposition 4.1 there is a skew-symmetric polarity of $F\mathbf{P}^3$, D for which:

$$D : V_i \mapsto span(V_{i+2}, V_{i+3}, V_{i+4})$$

with $i = 1, \ldots, 6$ and addition taken modulo 6. Now notice that D swaps opposite edges of our hexagon. For instance:

$$D(\overline{V_1V_2}) = D(V_1) \cap D(V_2) = span(V_3, V_4, V_5) \cap span(V_4, V_5, V_6) = \overline{V_4V_5}$$

Choose a plane $\mathcal{P}$ transverse to the edges of our hexagon. Then consider the line $\overline{A_1B_1}$ where $A_1 = \overline{V_1V_2} \cap \mathcal{P}$ and $B_1 = \overline{V_4V_5} \cap \mathcal{P}$. This line passes through both the lines $\overline{V_1V_2}$ and $\overline{V_4V_5}$ which are swapped by D, thus by proposition 4.2 the line $\overline{A_1B_1}$ is preserved by D. Similarly, $\overline{A_2B_2}$ and $\overline{A_3B_3}$ are preserved. Since $span(\overline{A_1B_1}, \overline{A_2B_2}, \overline{A_3B_3}) = \mathcal{P}$, we know that

$$D(\mathcal{P}) = \overline{A_1B_1} \cap \overline{A_2B_2} \cap \overline{A_3B_3}$$

Therefore these three lines are concurrent and intersect at the point $D(\mathcal{P})$. By duality the second half of the theorem follows. □

Remark 5. *If we project a skew hexagon into a plane it projects to a configuration ripe for us to apply Pappus' theorem to. As our polarity is skew-symmetric, we know that the dual to the point of projection is a plane containing that point. The projection of this plane is a line, which by construction must contain the points of intersection of the opposite sides of the hexagon. This is Pappus' theorem. Finally, as any Pappian configuration can be seen as a projection of a skew hexagon, we see that in fact the skew hexagon theorem implies Pappus' theorem.*

By equation (16), a direct application of the skew hexagon theorem tells us

$$\Pi_x^y(\ell^*(z_{even}, z_{odd})) = D(\Pi_x^y(F\mathbf{P}^2)) \tag{18}$$

Consequently, we will study this duality in much greater detail, especially in the lemma below. But before this, we need another seemingly irrelevant proposition. Recall at the start of this section we introduced the map $x \mapsto \rho_x$ from triples of points on x_l to representations of S_3 into $PGL(2,F)$. We never discussed any of this map's properties, but here is an important one:

Proposition 4.4 (Commuting Involution)**.** *Given a faithful representation $\rho : S_3 \to PGL(2,F)$, there are exactly two triples of points $x \neq x'$ such that $\rho_x = \rho = \rho_{x'}$. Moreover the projective transformation sending x to x' is an involution which commutes with $\rho(\upsilon)$ for all $\upsilon \in S_3$.*

Proof. First we will find x and x' from ρ. Since $\tau = (23)$, if $\rho_x = \rho$, it must be true that $\rho(\tau)(X_1) = X_1$. An involution has exactly two fixed points, so set X_1 and X_1' to be the two fixed points of $\rho(\tau)$. After setting X_1 in this manner, X_2 and X_3 are forced by $\sigma = (123)$ to be $X_2 = \rho(\sigma)(X_1)$ and $X_3 = \rho(\sigma^2)(X_1)$. Similarly X_2' and X_3' are forced.

We will check that these choices ensure $\rho_x = \rho$. Since $\rho(\sigma)(X_3) = X_1$ we see that $\rho_x(\sigma)$ agrees with $\rho(\sigma)$ at three points and therefore $\rho_x(\sigma) = \rho(\sigma)$. Also

$$\rho(\tau)(X_2) = \rho(\tau)\rho(\sigma)(X_1) = \rho(\sigma^2)\rho(\tau)(X_1) = X_3$$

And since $\rho_x(\tau)$ is an involution, $\rho(\tau)(X_3) = X_2$. Again $\rho_x(\tau)$ and $\rho(\tau)$ agree at three points and are equal. This shows ρ_x and ρ agree for generators $\sigma, \tau \in S_3$, consequently $\rho_x = \rho$. Similarly it follows that $\rho_{x'} = \rho$. Uniqueness of x and x' follows by the rigidity of process by which we selected X_1 and X_1'.

If $\rho = \rho_x = \rho_{x'}$, then the unique projective transformation $J : x \mapsto x'$ must commute with all of ρ since

$$\rho(\upsilon^{-1}) \circ J \circ \rho(\upsilon)(X_i) = \rho(\upsilon^{-1}) \circ J(X_{\upsilon(i)}) = \rho(\upsilon^{-1})(X'_{\upsilon(i)}) = X_i' = J(X_i)$$

for all i. Thus by the 3-transitivity of $PGL(2,F)$, $J = \rho(\upsilon^{-1}) \circ J \circ \rho(\upsilon)$ and so J and $\rho(\upsilon)$ commute. Because they commute, it follows that $\rho_{J(x')} = \rho$. Therefore by the uniqueness of the triples x and x' with the property that $\rho_x = \rho$, we know $J(x') = x$ or $J(x') = x'$. But the second possibility contradicts the fact that J is not the identity, therefore $J(x') = x$ and so J is an involution. □

The above proposition tells us that J is a natural invariant of a representation ρ, but we will find it easier notationally to associate the involution J with a triple of points, so J_x will now refer to this involution which commutes with all of ρ_x.

Now we provide a connection between the symmetric polarity $D_\mathcal{Q}$ described in proposition 3.1 and the skew symmetric polarities of $\mathbf{M}$.

Proposition 4.5. *If γ is an involution then $D_\mathcal{Q} \circ (\gamma\cdot)$ is a skew-symmetric polarity.*

Proof. If γ is an involution, then because of remark 1 below proposition 3.1, we know that $(\gamma\cdot)$ commutes with $D_\mathcal{Q}$ so

$$(D_\mathcal{Q} \circ (\gamma\cdot))^2 = D_\mathcal{Q} \circ (\gamma\cdot) \circ D_\mathcal{Q} \circ (\gamma\cdot) = D_\mathcal{Q} \circ D_\mathcal{Q} \circ (\gamma\cdot) \circ (\gamma\cdot) = D_\mathcal{Q}^2$$

Therefore $(\gamma\cdot) \circ D_\mathcal{Q}$ is a polarity. Since γ is an involution, $trace(\gamma) = 0$. Now recalling $dot_\mathbf{M}(N, M) = trace(M \cdot {}^\mathrm{T}N)$, we see that for $M \in \mathbf{M}$,

$$dot_\mathbf{M}(D_\mathcal{Q} \circ (\gamma\cdot)(M), M) = dot_\mathbf{M}({}^\mathrm{T}(\gamma M)^{-1}, M) = trace(M(\gamma M)^{-1}) = trace(\gamma) = 0$$

Thus $M \in D_{\mathcal{Q}} \circ (\gamma\cdot)(M)$ for all M, and $(\gamma\cdot) \circ D_{\mathcal{Q}}$ is a skew-symmetric polarity. □

Now we provide an alternate definition of the skew-symmetric polarity D in terms of the symmetric polarity $D_{\mathcal{Q}}$ described in proposition 3.1.

Lemma 4.6 (Polarity Identity)**.** *The duality D is the same as $D_{\mathcal{Q}} \circ (J_x\cdot) \circ (\rho_x(\tau)\cdot)$, where J_x is the involution appearing in proposition 4.4.*

Proof. We will denote $D_{\mathcal{Q}} \circ (J_x\cdot) \circ (\rho_x(\tau)\cdot)$ by $\hat{D}$. We know since J_x and $\rho_x(\tau)$ commute that $(J_x\cdot) \circ (\rho_x(\tau))$ is an involution, therefore by proposition 4.5, $\hat{D}$ is a skew-symmetric polarity. Next we will show that D and $\hat{D}$ both act on our skew hexagon in the same way, then by the skew hexagon theorem (theorem 4.3) we know that this information determines the skew-symmetric polarity via an explicit construction, so $D = \hat{D}$.

Now recall that $H_\upsilon = {}^{\mathrm{T}}\rho_x(\upsilon^{-1})$ from equation (12). First we will show $\hat{D}(H_\tau) = V_e = H_e \cap H_{\tau\sigma} \cap H_{\tau\sigma^2}$. Recall by remark 1, $\hat{D} = (J_x\cdot) \circ (\rho_x(\tau)\cdot) \circ D_{\mathcal{Q}}$.

$$\hat{D}(H_\tau) = (J_x\cdot) \circ (\rho_x(\tau)\cdot) \circ D_{\mathcal{Q}} = (J_x\cdot) \circ (\rho_x(\tau)\cdot)(\rho_x(\tau)) = J_x$$

So we must show that $J_x = V_e = H_e \cap H_{\tau\sigma} \cap H_{\tau\sigma^2}$. Since J_x is an involution, $J_x \in H_e = \mathsf{Tr}_0$. It remains to show that $J_x \in H_{\tau\sigma}$ and $J_x \in H_{\tau\sigma^2}$. Recall that J_x commutes with $\rho_x(\upsilon)$ for all $\upsilon \in S_3$.

We will show that if $T \in PGL(2, F)$ is an involution commuting with J_x then $dot_{\mathbf{M}}({}^{\mathrm{T}}T, J_x) = 0$. We again apply the identity $dot_{\mathbf{M}}(N, M) = trace({}^{\mathrm{T}}N \cdot M)$. Here we have $dot_{\mathbf{M}}({}^{\mathrm{T}}T, J_x) = trace(T \cdot J_x) = 0$, since $T \cdot J_x$ is an involution.

Thus since $V_e = J_x$ commutes with the involution $\rho_x(\tau\sigma)$, $dot_{\mathbf{M}}({}^{\mathrm{T}}\rho_x(\tau\sigma), J_x) = 0$ and so $J_x \in H_{\tau\sigma} = {}^{\mathrm{T}}\rho_x(\tau\sigma)$. Similarly $J_x \in H_{\tau\sigma^2}$, so we have shown $V_e = J_x$ and $\hat{D}(H_\tau) = D(H_\tau)$.

Now we wish to show that $V_\upsilon = J_x \cdot \rho_x(\upsilon)$. As $(\cdot\rho_x(\upsilon)) : H_{\upsilon'} \mapsto H_{\upsilon'\upsilon}$, it follows

$$(\cdot\rho_x(\upsilon))(V_e) = (\cdot\rho_x(\upsilon))(H_e \cap H_{\tau\sigma} \cap H_{\tau\sigma^2}) = H_\upsilon \cap H_{\tau\sigma\upsilon} \cap H_{\tau\sigma^2\upsilon} = V_\upsilon$$

Therefore $V_\upsilon = J_x \cdot \rho_x(\upsilon)$ as claimed. We now show $\hat{D}(H_\upsilon) = V_{\tau\upsilon}$ as expected:

$$\hat{D}(H_\upsilon) = (J_x\cdot) \circ (\rho_x(\tau)\cdot) \circ D_{\mathcal{Q}}({}^{\mathrm{T}}\rho_x(\upsilon^{-1})) = (J_x\cdot) \circ (\rho_x(\tau\upsilon)) = J_x \cdot \rho_x(\tau\upsilon) = V_{\tau\upsilon}$$

Consequently we have shown that $\hat{D}$ and D agree in their images of H_υ, and since both D and $\hat{D}$ are polarities, they also agree in their images of H_υ. By the skew hexagon theorem, skew-symmetric polarities which agree on a skew hexagon are equal, therefore $D = \hat{D}$. □

Now that we have a more useful form of D we can quickly prove:

Proof of Steiner's theorem II. We know $\Pi_x^y(\ell^*(z_{even}, z_{odd})) = D(\Pi_x^y(F\mathbf{P}^2))$ from equation (18). Also $D_{\mathcal{Q}}(\Pi_x^y(F\mathbf{P}^2)) = \Pi_x^y(O)$ where $O = x_l \cap y_l$ since $D_{\mathcal{Q}}$ swaps a point on the determinantal quadric with a plane tangent to the quadric, see theorem 2.1 and proposition 3.1. From lemma 4.6, we see that

$$\Pi_x^y(\ell^*(z_{even}, z_{odd})) = J_x \cdot \rho_x(\tau) \cdot \Pi_x^y(O) \tag{19}$$

Now recalling proposition 3.4 which told us that points on $\Pi_x^y(x_l)$ are characterized by the fact that they all have the same kernel. Thus

$$ker(\Pi_x^y(\ell^*(z_{even}, z_{odd}))) = ker(J_x \cdot \rho_x(\tau) \cdot \Pi_x^y(O)) = ker(\Pi_x^y(O))$$

which implies $\Pi_x^y(\ell^*(z_{even}, z_{odd}))) \in \Pi_x^y(x_l)$ and therefore $\ell^*(z_{even}, z_{odd}) \in x_l$.

By substituting σx or $\sigma^2 x$ for x we get see that the same argument shows that $\ell^*(z_{even}, \sigma z_{odd})$ and $\ell^*(z_{even}, \sigma^2 z_{odd})$ lie on x_l. More specifically, if we let $x' = \sigma x$ then and defined z'_{even} and z'_{odd} accordingly, we see that:

$$\begin{aligned} z'_{even} &= (\ell(x', y), \ell(x', \sigma y), \ell(x', \sigma^2 y)) &&= (\ell(\sigma x, y), \ell(\sigma x, \sigma y), \ell(\sigma x, \sigma^2 y)) \\ &= (\ell(x, \sigma^2 y), \ell(x, y), \ell(x, \sigma y)) &&= \sigma z_{even} \end{aligned}$$

And,

$$\begin{aligned} z'_{odd} &= (\ell(x', \tau y), \ell(x', \tau\sigma y), \ell(x', \tau\sigma^2 y)) &&= (\ell(\sigma x, \tau y), \ell(\sigma x, \tau\sigma y), \ell(\sigma x, \tau\sigma^2 y)) \\ &= (\ell(x, \tau\sigma y), \ell(x, \tau\sigma^2 y), \ell(x, \tau y)) &&= \sigma^2 z_{odd} \end{aligned}$$

Thus,

$$\ell^*(z'_{even}, z'_{odd}) = \ell^*(\sigma z_{even}, \sigma^2 z_{odd}) = \ell^*(z_{even}, \sigma z_{odd}) \tag{20}$$

Therefore $\ell^*(z_{even}, \sigma z_{odd})$ lies on x_l as well. Similarly by substitution of $\sigma^2 x$ for x shows us that $\ell^*(z_{even}, \sigma^2 z_{odd})$ lie on x_l as well.

We still must show that $\ell^*(z_{even}, \upsilon z_{odd})$ returns to the line y_l when υ is odd. We will prove this by investigating what happens when we switch x with y. For this we define $x' = y$ and $y' = x$. Then we define z'_{even} and z'_{odd} accordingly. We see that since $\ell(y, x) = \ell(x, y)$,

$$\begin{aligned} z'_{even} &= (\ell(x', y'), \ell(x', \sigma y'), \ell(x', \sigma^2 y')) &&= (\ell(y, x), \ell(y, \sigma x), \ell(y, \sigma^2 x)) \\ &= (\ell(x, y), \ell(x, \sigma^2 y), \ell(x, \sigma y)) &&= \tau z_{even} \end{aligned}$$

and for z'_{odd} we see

$$\begin{aligned} z'_{odd} &= (\ell(x', \tau y'), \ell(x', \tau\sigma y'), \ell(x', \tau\sigma^2 y')) &&= (\ell(y, \tau x), \ell(y, \tau\sigma x), \ell(y, \tau\sigma^2 x)) \\ &= (\ell(x, \tau y), \ell(x, \tau\sigma y), \ell(x, \tau\sigma^2 y)) &&= z_{odd} \end{aligned}$$

Therefore

$$\ell^*(z'_{even}, z'_{odd}) = \ell^*(\tau z_{even}, z_{odd}) = \ell^*(z_{even}, \tau z_{odd})$$

which implies $\ell^*(z_{even}, \tau z_{odd}) \in x'_l = y_l$. Similarly, we can see $\ell^*(z_{even}, \tau\sigma z_{odd})$ and $\ell^*(z_{even}, \tau\sigma^2 z_{odd})$ lie on y_l. □

We have finally developed enough machinery that the final part of Steiner's theorem falls easily.

Proof of Steiner's theorem III. Proposition 3.4 tells us that $X \in x_l$ implies that $img(\Pi_x^y(X)) = X$. Applying this fact to equation (19) above, we see that

$$\ell^*(z_{even}, z_{odd}) = img(J_x \cdot \rho_x(\tau) \cdot \Pi_x^y(O)) = J_x \cdot \rho_x(\tau) \cdot img(\Pi_x^y(O)) = J_x \cdot \rho_x(\tau)(O) \tag{21}$$

In the previous proof, when we set $x' = \sigma x$ we saw equation (20). Therefore, by applying (21), we see

$$\ell^*(z_{even}, \sigma z_{odd}) = \ell^*(z'_{even}, z'_{odd}) = J_{x'} \cdot \rho_{x'}(\tau)(O) = J_{\sigma x} \cdot \rho_{\sigma x}(\tau)(O)$$

Now $\rho_{\sigma x}(\upsilon) = \rho_x(\sigma^2 \upsilon \sigma)$, because $\rho_x(\sigma)(x) = \sigma^{-1} x$ as defined in equation (11), and therefore $J_x = J_{\sigma x}$ and so

$$\ell^*(z_{even}, \sigma z_{odd}) = J_{\sigma x} \cdot \rho_{\sigma x}(\tau)(O) = J_x \cdot \rho_x(\tau\sigma^2)(O) \tag{22}$$

Similarly, it follows that

$$\ell^*(z_{even}, \sigma^2 z_{odd}) = J_x \cdot \rho_x(\tau\sigma)(O) \tag{23}$$

Now by equations (21), (22), and (23) we know that $\ell^*(z_{even}, \upsilon z_{odd})$ can be determined from J_x, ρ_x, and O. This information is independent from y except for the fact that $O = y_l \cap x_l$. This proves the theorem. □

We define the Steiner map to be the return map $\mathcal{S}_O : (F\mathbf{P}^1)^3 \to (F\mathbf{P}^1)^3$, such that

$$\mathcal{S}_O : x \mapsto (\ell^*(z_{even}, z_{odd}), \ell^*(z_{even}, \sigma^2 z_{odd}), \ell^*(z_{even}, \sigma z_{odd}))$$

Where the points $\ell^*(z_{even}, \upsilon z_{odd})$ are constructed as before from x and any triple of points y which are colinear with the point O. Now we accumulate a few corollaries from our calculations. First from equations (21), (22), and (23) we have:

Corollary 4.7. $\mathcal{S}_O(x) = (J_x \cdot \rho_x(\tau)(O), J_x \cdot \rho_x(\tau\sigma)(O), J_x \cdot \rho_x(\tau\sigma^2)(O))$

Corollary 4.8 (Permutations commute)**.** *Permutations commute with $\mathcal{S}_O$. However $\rho_x(\sigma)$ anti-commutes with $\mathcal{S}_O$, that is $\mathcal{S}_O \circ \rho_x(\sigma)(x) = \rho_x(\sigma^2) \circ \mathcal{S}_O(x)$.*

Proof. We will show that the generators $\sigma, \tau \in S_3$ commute with $\mathcal{S}_O$. We need to understand how permuting x effects ρ_x. We can see $\rho_{\upsilon x}(\upsilon') = \rho_x(\upsilon^{-1}\upsilon'\upsilon)$. Therefore

$$\mathcal{S}_O(\sigma x) = (J_x \cdot \rho_x(\tau\sigma^2)(O), J_x \cdot \rho_x(\tau)(O), J_x \cdot \rho_x(\tau\sigma)(O)) = \sigma\mathcal{S}_O(x)$$
$$\mathcal{S}_O(\tau x) = (J_x \cdot \rho_x(\tau)(O), J_x \cdot \rho_x(\tau\sigma^2)(O), J_x \cdot \rho_x(\tau\sigma)(O)) = \tau\mathcal{S}_O(x)$$

Finally to show $\rho_x(\sigma)$ anti-commutes with $\mathcal{S}_O$, we show $\rho_x(\sigma)(\mathcal{S}_O(x)) = \sigma\mathcal{S}_O(x)$.

$$\begin{aligned} \rho_x(\sigma)(\mathcal{S}_O(x)) &= \rho_x(\sigma)(J_x \cdot \rho_x(\tau)(O), J_x \cdot \rho_x(\tau\sigma^2)(O), J_x \cdot \rho_x(\tau\sigma)(O)) \\ &= (J_x \cdot \rho_x(\tau\sigma^2)(O), J_x \cdot \rho_x(\tau\sigma)(O), J_x \cdot \rho_x(\tau)(O)) = \sigma\mathcal{S}_O(x) \end{aligned}$$

Now because σ commutes, $\rho_x(\sigma)(\mathcal{S}_O(x)) = \mathcal{S}_O(\sigma x) = \mathcal{S}_O(\rho_x(\sigma^2)(x))$, again by recalling the definition of ρ_x from equation (11). □

Corollary 4.9 (Steiner's Map is 2-1)**.** *$\mathcal{S}_O(x) = \mathcal{S}_O(x')$ if and only if $\rho_x = \rho_{x'}$. Thus the map $\mathcal{S}_O$ is two to one, with $\mathcal{S}_O(x) = \mathcal{S}_O(x')$ if and only if $J_x(x) = x'$.*

Proof. It is clear that $\mathcal{S}_O(x) = \mathcal{S}_O(J_x(x))$, since corollary 4.7 determines $\mathcal{S}_O(x)$ from O and ρ_x only. We claim that there is no other x' such that $\mathcal{S}_O(x) = \mathcal{S}_O(x')$.

Now suppose $\mathcal{S}_O(x) = \mathcal{S}_O(x') = (A, B, C) \in (F\mathbf{P}^1)^3$. Then $\rho_x(\sigma)(A) = C$, $\rho_x(\sigma)(B) = A$ and $\rho_x(\sigma)(C) = B$ and similarly for x'. Therefore $\rho_x(\sigma) = \rho_{x'}(\sigma)$.

Given a representation $\rho : S_3 \mapsto PGL(2, F)$, there is a related representation, which we will call ρ' and define as

$$\rho'(\sigma) = \rho(\sigma) \quad \text{and} \quad \rho'(\tau) = J_\rho \cdot \rho(\tau)$$

where J_ρ is determined by ρ as in lemma 4.6. This can easily be checked to be a representation. Also note that J_ρ commutes with each $\rho'_x(\upsilon)$, therefore $J_{\rho'} = J_\rho$ and so $(\rho')' = \rho$.

Set $D_1 = O = x_l \cap y_l$, $D_2 = \rho_x(\sigma)(O)$ and $D_3 = \rho_x(\sigma^2)(O)$ which would be defined in the same way for x' since $\rho_x(\sigma) = \rho_{x'}(\sigma)$. Now by corollary 4.7,

$$\rho'_x(\tau)(D_i) = \rho'_{x'}(\tau)(D_i) = (\mathcal{S}_O(x))_i$$

for each i. Thus $\rho'_x(\tau) = \rho'_{x'}(\tau)$ and so the representations ρ'_x and $\rho'_{x'}$ are equal. Consequently $(\rho'_x)' = (\rho'_{x'})'$ which implies $\rho_x = \rho_{x'}$. It follows then by lemma 4.4 that $\mathcal{S}_O$ is 2-1. □

Corollary 4.10. *$\mathcal{S}_O$ permutes triples x which together with $O = x_l \cap y_l$ form a harmonic quadruple (a quadruple having cross ratio -1). If x' is a triple of points such that one of its points is O then $\mathcal{S}_O(x')$ together with O form a harmonic quadruple.*

Proof. We prove the first statement. Up to permutation of x and $PGL(2, F)$, a harmonic quadruple must be $O = {}^{\mathrm{T}}(1:0)$ and $x = ({}^{\mathrm{T}}(0:1), {}^{\mathrm{T}}(1:1), {}^{\mathrm{T}}(-1:1))$. We can compute

$$\rho_x(\tau) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \quad \rho_x(\sigma) = \begin{pmatrix} 1 & 1 \\ -3 & 1 \end{pmatrix} \quad J_x = \begin{pmatrix} 0 & 1 \\ 3 & 0 \end{pmatrix}$$

which can be checked by ensuring $\rho_x(\upsilon)(X_i) = X_{\upsilon(i)}$ and J_x commutes with all $\rho_x(\upsilon)$. Then by corollary 4.7, we compute $\mathcal{S}_O(x) = ({}^{\mathrm{T}}(0:1), {}^{\mathrm{T}}(-1:1), {}^{\mathrm{T}}(1:1)) = \tau x$.

Next notice that $J_x(x) = ({}^{\mathrm{T}}(1:0), {}^{\mathrm{T}}(3:1), {}^{\mathrm{T}}(-3:1))$, so in particular $O \in J_x(x)$. And by corollary 4.9, we know that $\mathcal{S}_O(J_x(x)) = \mathcal{S}_O(x) = \tau x$, a triple of points which together with O form a harmonic quadruple. Now since any three points in the plane differ only by a projective transformation, the three points of $J_x(x)$ and O, are projectively equivalent to any other triple x' containing O modulo permutation. Thus by Steiner's theorem III, x' and O determine $\mathcal{S}_O(x')$, so for any triple of points x' containing O it must be true that $\mathcal{S}_O(x')$ together with O form a harmonic quadruple. □

These corollaries have given us a lot of useful information about the dynamics of $\mathcal{S}_O$. In the next section, we will do an explicit calculation, which will give us much stronger results.

5. Dynamics and The Twisted Cubic

In this section we will denote the set of orderless triples of points in $F\mathbf{P}^1$ by $\mathcal{T}$. Because $\mathcal{S}_O$ commutes with permutation, it makes sense to consider instead the map $\hat{\mathcal{S}}_O : \mathcal{T} \to \mathcal{T}$ which we just define as the map that "forgets" order. Explicitly we can describe $\hat{\mathcal{S}}_O$ as the map which takes a triple and gives it an arbitrary ordering, then applies the map $\mathcal{S}_O$, and then forgets the ordering. This map is well defined since $\mathcal{S}_O$ commutes with permutation. Also we will frequently use the identification of $F\mathbf{P}^1$ with $F \cup \infty$, which turns the matrices of $PGL(2, F)$ into fractional linear transformations of the form $z \mapsto \frac{az+b}{cz+d}$. This makes our equations nicer, just as it does in complex analysis.

Over algebraically closed fields, $\mathcal{T}$ is canonically isomorphic to $F\mathbf{P}^3$. To see this, consider the map which takes three points to the homogeneous polynomial of degree three with the three points as roots. The space of such polynomials modulo scalar multiplication is $F\mathbf{P}^3$. Explicitly, our map is:

$$\Sigma : \{a, b, c\} \mapsto (a + b + c, ab + ac + bc, abc) \tag{24}$$

and projectivizes by setting $a = \frac{a_1}{a_0}$, $b = \frac{b_1}{b_0}$, and $c = \frac{c_1}{c_0}$

$$\Sigma : \{(a_0 : a_1), (b_0 : b_1), (c_0 : c_1)\} \mapsto \begin{array}{l} \{\, a_0b_0c_0 : a_1b_0c_0 + a_0b_1c_0 + a_0b_0c_1 : \\ \;\; a_1b_1c_0 + a_1b_0c_1 + a_0b_1c_1 : a_1b_1c_1 \,\} \end{array}$$

The map is tri-linear, that is, linear in each of the points. Thus for example, image of the space of all triples of points containing a particular point P is a planar subset of $\mathcal{P}_P \subset F\mathbf{P}^3$. We will be considering the map $\Sigma \circ \hat{\mathcal{S}}_O \circ \Sigma^{-1}$ in order to gain a better understanding of the global properties of the map $\hat{\mathcal{S}}_O$.

The image of all triples for which all three points are equal is the twisted cubic

$$\mathbf{T} = \{\Sigma(A, A, A) | A \in F\mathbf{P}^1\} = \{(a_0^3 : 3a_0^2 a_1 : 3a_0 a_1^2 : a_1^3) | (a_0 : a_1) \in F\mathbf{P}^1\} \tag{25}$$

A twisted cubic is a 1-dimensional variety of degree 3 with many beautiful properties. In particular, for every point in $F\mathbf{P}^3 \smallsetminus \mathbf{T}$ there is a unique secant (a line passing through $\mathbf{T}$ at two points) or tangent which passes through that point. A weaker version of this fact follows from lemma 5.2 below, but first we have

Proposition 5.1. *Given $\pi \in PGL(2, F)$, there is a $\pi' \in PGL(4, F)$ such that $\Sigma(\pi A, \pi B, \pi C) = \pi' \Sigma(A, B, C)$ for all $A, B, C \in F\mathbf{P}^1$. Moreover, such a π' preserves $\mathbf{T}$.*

Proof. This follows from the interpretation of Σ as the projectivization of symmetric polynomials since each coordinate of $\Sigma(\pi A, \pi B, \pi C)$ will be a symmetric polynomial of degree 3 and therefore a linear sum of the coordinates of $\Sigma(A, B, C)$. Also such a π' must preserve $\mathbf{T}$, since the map $(A, B, C) \mapsto (\pi A, \pi B, \pi C)$ preserves the set of triples consisting of three identical points. □

If $x \in \mathcal{T}$, there is some ambiguity in the definition of ρ_x. In particular ρ_x is only defined up to inner automorphisms of S_3. We define the "relation" $a \sim b$ when $a \in \mathcal{T}$ is a triple of distinct points and $b \in \mathcal{T}$ as $a \sim b$ if and only if $\rho_a(\sigma)(b) = b$ as an orderless triple. This is well defined since if $\rho_a(\sigma)(b) = b$ then also $\rho_a(\sigma^2)(b) = b$. Now we define the set $\mathcal{O}_x = \{x' \in \mathcal{T} | x \sim x'\}$. This set is relevant because corollary 4.8 told us that $\mathcal{S}_O$ anti-commutes with $\rho_x(\sigma)$, therefore the orbit of x under $\hat{\mathcal{S}}_O$ lies in $\mathcal{O}_x$. More concisely, the set $\mathcal{O}_x$ is invariant under the action of $\hat{\mathcal{S}}_O$.

First, we are interested in the image $\Sigma(\mathcal{O}_x)$.

Lemma 5.2. *$\Sigma(\mathcal{O}_x)$ is a secant of $\mathbf{T}$, that is $\Sigma(\mathcal{O}_x)$ is a line which passes through $\mathbf{T}$ at two points. Further, every secant arises as $\Sigma(\mathcal{O}_{x'})$ for some triple x' of distinct points.*

Proof. Fix x. We will first show that if we suppose $\Sigma(\mathcal{O}_x)$ is a secant then every secant arises from some $\Sigma(\mathcal{O}_{x'})$. Since $\Sigma(\mathcal{O}_x)$ is a secant, by definition it passes through two points on $\mathbf{T}$ say $\Sigma(f_1, f_1, f_1)$ and $\Sigma(f_2, f_2, f_2)$. Now choose an arbitrary secant l which must contain 2 points of $\mathbf{T}$ say $\Sigma(f'_1, f'_1, f'_1)$ and $\Sigma(f'_2, f'_2, f'_2)$. There is a $\pi \in PGL(2, F)$ such that $\pi(f_1) = f'_1$ and $\pi(f_2) = f'_2$. By proposition 5.1 above, then there is a $\pi' \in PGL(4, F)$ fixing $\mathbf{T}$ such that $\Sigma(\pi A, \pi B, \pi C) = \pi' \Sigma(A, B, C)$. For all distinct triples y, it is true that $\rho_{\pi y}(\upsilon) = \pi \circ \rho_y(\upsilon) \circ \pi^{-1}$ because

$$\rho_{\pi y}(\upsilon)(\pi X_i) = \pi \circ \rho_y(\upsilon) \circ \pi^{-1}(\pi X_i) = \pi \circ \rho_y(\upsilon)(X_i) = \pi X_{\upsilon(i)}$$

Thus $y \sim x$ implies $\pi y \sim \pi x$ and so $\mathcal{O}_{x'} = \pi \mathcal{O}_x$. Then if we set $x' = \pi x$, we have

$$\pi'(\Sigma(\mathcal{O}_x)) = \Sigma(\pi \mathcal{O}_x) = \Sigma(\mathcal{O}_{x'})$$

Moreover, since $\pi(f_i) = f'_i$ for each i we know that $\pi'(\Sigma(f_i, f_i, f_i)) = \Sigma(f'_i, f'_i, f'_i)$. Therefore if $\Sigma(\mathcal{O}_x)$ is a secant passing through $\Sigma(f_i, f_i, f_i)$ for each i then $\Sigma(\mathcal{O}_{x'}) = \pi'(\Sigma(\mathcal{O}_x))$ is a secant passing through each $\Sigma(f'_i, f'_i, f'_i)$ which must be l as desired.

So we only need to show $\Sigma(\mathcal{O}_x)$ is a secant of $\mathbf{T}$ for some x. This will be an explicit calculation. We will think of $F\mathbf{P}^1$ as $F \cap \{\infty\}$. First, as in the proof of proposition 4.1, by the 3-transitivity of $PGL(2, F)$ and because of the paragraph above we can choose x arbitrarily. We choose $x = (1, \omega, \omega^2)$ with ω a cube root of unity ($\omega^3 = 1$ but $\omega \neq 1$). This gives us that $\rho_x(\sigma) : z \mapsto \omega z$. Now if $y \sim x$ then $y = \{z, \rho_x(\sigma)(z), \rho_x(\sigma^2)(z)\}$ for some z therefore $y = \{z, \omega z, \omega^2 z\}$. Thus

$$\mathcal{O}_x = \{\{z, \omega z, \omega^2 z\} | z \in F \cup \{\infty\}\} \tag{26}$$

We compute the image of y under Σ using equation (24)

$$\begin{aligned} \Sigma(y) &= \Sigma(\{z, \omega z, \omega^2 z\}) = ((1+\omega+\omega^2)z, (\omega+\omega^2+1)z^2, z^3) \\ &= (0, 0, z^3) = (1 : 0 : 0 : z^3) \end{aligned} \tag{27}$$

Thus $\Sigma(\mathcal{O}_x) = \{(a : 0 : 0 : b) | (a : b) \in F\mathbf{P}^1\}$, which is a line in $F\mathbf{P}^3$. We must show that this line is a secant. 0 and ∞ are fixed points of $\rho_x(\sigma)$ therefore $\{0, 0, 0\}, \{\infty, \infty, \infty\} \in \mathcal{O}_x$ and therefore the line $\Sigma(\mathcal{O}_x)$ must pass through $\Sigma(\{0, 0, 0\})$ and $\Sigma(\{\infty, \infty, \infty\})$, and so $\Sigma(\mathcal{O}_x)$ is a secant line of $\mathbf{T}$. □

Remark 6 (Secants to the twisted cubic cover $F\mathbf{P}^3$)**.** *We have shown that $\Sigma(x)$ lies on a secant of $\mathbf{T}$ if x consists of distinct points. Since if x consists of three identical points $\Sigma(x) \in \mathbf{T}$, the only points left to check are of the form $x = \{A, A, B\}$. We can check that as we vary B the image $\Sigma(\{A, A, B\})$ varies in a line. More careful examination would reveal that $\Sigma(\{A, A, B\})$ lies on the tangent to $\mathbf{T}$ through $\Sigma(\{A, A, A\})$. This together with the fact that Σ is isomorphic to $\mathcal{T}$ would show that every point lies on a unique secant or tangent of $\mathbf{T}$.*

Proposition 5.3. *If $x = (1, \omega, \omega^2)$ and O is not fixed by $\rho_x(\sigma)$ then when restricted the line $l = \Sigma(\mathcal{O}_x)$ the function $\Sigma \circ \hat{\mathcal{S}}_O \circ \Sigma^{-1}$ acts as the map $z \mapsto z^2$ up to conjugation in $PGL(2, F)$ acting on l. Moreover, this conjugation identifies the points at which the secant line l passes through $\mathbf{T}$ with 0 and ∞.*

Proof. This proposition will continue the calculation of lemma 5.2 and use corollary 4.7 as the tool to obtain a formula. We will continue to think of $F\mathbf{P}^1$ as $F \cup \{\infty\}$.

In order to use corollary 4.7, we need to know the location $O = x_l \cap y_l \in F \cup \{\infty\}$. We will show that the map $\Sigma \circ \hat{\mathcal{S}}_O \circ \Sigma^{-1}$ is conjugate to the map $z \mapsto z^2$ with the desired properties by an explicit calculation. By reversing equation (27),

$$\Sigma \circ \hat{\mathcal{S}}_O \circ \Sigma^{-1}((0, 0, z)) = \Sigma \circ \hat{\mathcal{S}}_O(\{\sqrt[3]{z}, \omega\sqrt[3]{z}, \omega^2\sqrt[3]{z}\})$$

Notice the choice of $\sqrt[3]{z}$ is unimportant, since essentially this is a set of all $\sqrt[3]{z}$. Set $x' = \{\sqrt[3]{z}, \omega\sqrt[3]{z}, \omega^2\sqrt[3]{z}\}$. In order to apply 4.7, we need to know $\rho_{x'}$ and $J_{x'}$. Here are our candidates:

$$\rho_{x'}(\sigma) : t \mapsto \omega t \quad \rho_{x'}(\tau) : t \mapsto \tfrac{\sqrt[3]{z}^2}{t} \quad J_{x'} : t \mapsto -t \tag{28}$$

It can be checked that $\rho_{x'}(\sigma)$ and $\rho_{x'}(\tau)$ act as they should on x' and that $J_{x'}$ is an involution commuting with both $\rho_{x'}(\sigma)$ and $\rho_{x'}(\tau)$. Then

$$\hat{\mathcal{S}}_O(x') = (\frac{-\sqrt[3]{z}^2}{O}, \frac{-\omega^2\sqrt[3]{z}^2}{O}, \frac{-\omega\sqrt[3]{z}^2}{O})$$

So by a calculation

$$\Sigma \circ \hat{\mathcal{S}}_O(x') = (0, 0, \frac{-z^2}{O^3})$$

and

$$\Sigma \circ \hat{\mathcal{S}}_O \circ \Sigma^{-1}((0, 0, z)) = \Sigma \circ \hat{\mathcal{S}}_O(x') = (0, 0, \frac{-z^2}{O^3}) \tag{29}$$

Now to show this is conjugate to the map $z \mapsto z^2$ we introduce the projective embedding $\theta : F \cup \{\infty\} \to \Sigma(\mathcal{O}_x)$ defined as $\theta(w) = (0, 0, -O^3 w)$. Then

$$\begin{aligned} \theta^{-1} \circ \Sigma \circ \hat{\mathcal{S}}_O \circ \Sigma^{-1} \circ \theta(w) &= \theta^{-1} \circ \Sigma \circ \hat{\mathcal{S}}_O \circ \Sigma^{-1}((0, 0, -O^3 w)) \\ &= \theta^{-1}((0, 0, \tfrac{-(-O^3 w)^2}{O^3}) = \theta^{-1}((0, 0, -O^3 w^2)) = w^2 \end{aligned}$$

Thus this map is conjugate to $w \mapsto w^2$. We need to show that this act of conjugation sends 0 and ∞ to points $\Sigma(\mathcal{O}_x) \cap \mathbf{T}$. From the end of the proof of lemma 5.2 these fixed points are $\Sigma(\{0, 0, 0\}) = (0, 0, 0) = (1 : 0 : 0 : 0)$ and $\Sigma(\{\infty, \infty, \infty\}) = (0, 0, \infty) = (0 : 0 : 0 : 1)$. These points are sent to 0 and ∞ by θ. $\square$

Well, what happens when O is a fixed point of $\rho_x(\sigma)$? Well, equation (29) still holds and $O = 0$ or $O = \infty$, therefore

$$\Sigma \circ \hat{\mathcal{S}}_O \circ \Sigma^{-1}((0, 0, z)) = (0, 0, \frac{-z^2}{O^3})$$

which means that the image of $\Sigma \circ \hat{\mathcal{S}}_O \circ \Sigma^{-1}$ is $\Sigma(\{F, F, F\})$ where F is the fixed point of $\rho_x(\sigma)$ other than O.

Theorem 5.4 (Global action of $\mathcal{S}_O$)**.** *The map $\Sigma \circ \hat{\mathcal{S}}_O \circ \Sigma^{-1}$ preserves the secants to the twisted cubic $\mathbf{T}$ which do not pass through $\Sigma(\{O, O, O\})$. The action of this map when restricted to such a secant line l is the map $z \mapsto z^2$ up to conjugation in $PGL(2, F)$, the group of projective transformations l. This conjugation sends the points at which the secant passes through the $\mathbf{T}$ to 0 and ∞, and sends the point $l \cap \mathcal{P}_O$ to -1, determining this conjugation. Recall $\mathcal{P}_O$ is the plane consisting of the image under Σ of the set of all triples containing O.*

Proof. If l is a secant not passing through $\Sigma(\{O, O, O\})$ then analogously to the proof of lemma 5.2, we will show the choice of such an l does not matter.

First if l does not pass through $\Sigma(\{O, O, O\})$ then it does not lie in the plane $\mathcal{P}_O$, since the points at which it cross $\mathbf{T}$ cannot lie in $\mathcal{P}_O$, because these points consist of three identical points which cannot be O.

Now, if l and l' are secants passing through $\mathbf{T}$ at points $\Sigma(\{f_i, f_i, f_i\}$ and $\Sigma(\{f'_i, f'_i, f'_i\}$ respectively for $i = 1, 2$. We can choose $\pi \in PGL(2, F)$ preserving O and mapping $f_i \mapsto f'_i$. The map π commutes with $\hat{\mathcal{S}}_O$, since $\hat{\mathcal{S}}_O$ is determined by O which is preserved by π. Then by proposition 5.1, there is a π' for which $\pi'(\Sigma(A, B, C)) = \Sigma(\pi A, \pi B, \pi C)$. In particular then π' commutes with $\Sigma \circ \hat{\mathcal{S}}_O \circ \Sigma^{-1}$ because π commutes with $\hat{\mathcal{S}}_O$. Also $\pi'(\Sigma(\{f_i, f_i, f_i\})) = \Sigma(\{f'_i, f'_i, f'_i\})$ for each i, and therefore $\pi' : l \mapsto l'$. Notice that also

$$\pi'(l \cap \mathcal{P}_O) = \pi'(l) \cap \pi(\mathcal{P}_O) = l' \cap \mathcal{P}_O$$

since $\pi : O \mapsto O$ and thus preserves the image of the set of all triples containing O, which therefore by definition of π' implies that π' must preserve $\mathcal{P}_O$. Therefore if the action of $\Sigma \circ \hat{\mathcal{S}}_O \circ \Sigma^{-1}$ on a secant l is determined by the points $l \cap \mathbf{T}$ and $l \cap \mathcal{P}_O$ as claimed by the theorem, if the statement is true for l then it is true for l', because the points $l \cap \mathbf{T}$ and $l \cap \mathcal{P}_O$ are sent to $l' \cap \mathbf{T}$ and $l' \cap \mathcal{P}_O$ by π' which commutes with $\Sigma \circ \hat{\mathcal{S}}_O \circ \Sigma^{-1}$.

Without loss of generality, assume O is not 0 or ∞. Then proposition 5.3 told us that the action of $\Sigma \circ \mathcal{S}_O \circ \Sigma^{-1}$ on $l = \Sigma(\mathcal{O}_x)$ is $z \mapsto z^2$ up to conjugation in $PGL(2, F)$. Moreover, we know that the points of $l \cap \mathbf{T}$ serve the roll of 0 and ∞ in the map $z \mapsto z^2$. Consider the triple of points $\{O, A, B\}$ such that

$\Sigma(\{O,A,B\}) = l \cap \mathcal{P}_O$. By corollary 4.10, we know that $H = \hat{\mathcal{S}}_O(\{O,A,B\})$ is a triple of points harmonic to 0 and that $\hat{\mathcal{S}}_O(H) = H$. The map $z \mapsto z^2$, has 3 fixed points, 0, 1, and ∞. Therefore since the points $l \cap \mathbf{T}$ correspond to the other two of them, $\Sigma(H)$ must correspond to 1. Then $\Sigma(\{O,A,B\}) = l \cap \mathcal{P}_O$ must correspond to -1. Therefore $\Sigma \circ \hat{\mathcal{S}}_O \circ \Sigma^{-1}$ acts as claimed on $l = \Sigma(\mathcal{O}_x)$. The preceding paragraph then tells us that it acts as claimed on any line l' secant to $\mathbf{T}$ and not passing through $\Sigma(\{O,O,O\})$. □

We can explicitly compute the map $\Psi = \Sigma \circ \hat{\mathcal{S}}_O \circ \Sigma^{-1}$ for $O = (0:1)$ as

$$\begin{aligned}
&\Psi((a:b:c:d)) = \\
&\quad (2b^6 - 18ab^4c + 27a^2b^2c^2 + 54a^3c^3 + 108a^2b^3d - 486a^3bcd + 729a^4d^2 : \\
&\quad 3b^5c - 45ab^3c^2 + 135a^2bc^3 + 81ab^4d - 324a^2b^2cd - 243a^3c^2d + 729a^3bd^2 : \\
&\quad -3b^4c^2 + 54a^2c^4 + 18b^5d - 54ab^3cd - 162a^2bc^2d + 243a^2b^2d^2 : \\
&\quad -2b^3c^3 + 9abc^4 + 9b^4cd - 54ab^2c^2d + 27a^2c^3d + 27ab^3d^2)
\end{aligned}$$

Remark 7 (The Real Case). *In theorem 5.4 there is an explicit parameterization of a secant l of $\mathbf{T}$, namely the one sending $l \cap \mathbf{T}$ to 0 and ∞ and sending $l \cap \mathcal{P}_O$ to -1. It can be checked that if a triple has real cross ratio with respect to O, it is sent to the unit circle in this parameterization by Σ. Also an order three element of $PGL(2,\mathbb{R})$ has imaginary fixed points, so it will never happen that $\rho_x(\sigma)$ preserves $O = x_l \cap y_l$. Consequently, for all x, $\Sigma(\mathcal{O}_x)$ is canonically identified with the unit circle of $\mathcal{C}$ with $\Sigma \circ \hat{\mathcal{S}}_O \circ \Sigma^{-1}$ acting as $z \mapsto z^2$, a double wrapping action on $\Sigma(\mathcal{O}_x)$. In this case $\hat{\mathcal{S}}_O$ is genuinely $2-1$ and onto, and the only fixed points are triples of points harmonic to O.*

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