IMMERSIONS AND THE SPACE OF ALL TRANSLATION SURFACES

W. PATRICK HOOPER

This is an extended abstract for a talk given at the Mathematisches Forschungsinstitut Oberwolfach in the workshop on "Flat Surfaces and Dynamics on Moduli Space" held March 23-29, 2014.

For this talk, a *translation surface* is a topological surface equipped with an atlas of charts to the plane so that the transition function are translations. Note that this definition means that our surfaces have no singularities, and all translation surfaces other than quotients of the plane are incomplete. Our translation surfaces will be pointed, i.e., they come with a choice of a basepoint. Our goal is to place a topology on the space $\mathcal{M}$ of all translation surfaces, which includes incomplete surfaces of infinite topological type, and to draw connections to associated dynamical systems.

Immersions

Let D be a simply connected subset of a translation surface containing the basepoint. An *immersion* of D into a (pointed) translation surface S is a continuous map $D \rightsquigarrow S$ which sends the basepoint of D to the basepoint of S, and which acts as a translation in local coordinates.

We remark that the restriction of the notion of immersion to the space $\tilde{\mathcal{M}}$ of all pointed simply connected translation surfaces yields a partial ordering on $\tilde{\mathcal{M}}$. The following result discusses the structure of this ordering.

Theorem 1. *Let* 0 *denote the degenerate translation surface consisting of a single point; it immerses in everything. The set* $\tilde{\mathcal{M}} \cup \{0\}$ *equipped with the partial order* $\rightsquigarrow$ *forms a complete lattice, i.e., each subset of* $\tilde{\mathcal{M}} \cup \{0\}$ *has a supremum and an infimum in* $\tilde{\mathcal{M}} \cup \{0\}$.

Topologies on moduli spaces

We use immersions to define the *immersive topology* on $\tilde{\mathcal{M}}$. A sequence of simply connected translation surfaces $\tilde{S}_n \in \tilde{\mathcal{M}}$ converges to $\tilde{S} \in \tilde{\mathcal{M}}$ if the following two statements hold:

- For every $K \subset \tilde{S}$ homeomorphic to a closed disk which contains the basepoint in its interior, there is an N so that $K \rightsquigarrow \tilde{S}_n$ for $n > N$.
- For every $\tilde{U} \in \tilde{\mathcal{M}}$, if $\tilde{U} \rightsquigarrow \tilde{S}_n$ for infinitely many n, then $\tilde{U} \rightsquigarrow \tilde{S}$.

(We just state the notion of convergence of sequences here because sequences are more relevant to this discussion, but each statement above corresponds to collection of open sets. See [Hoo13b] for a formal definition of the topology.)

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There is a canonical disk bundle over $\tilde{\mathcal{E}} \to \tilde{\mathcal{M}}$, where the fiber over a surface $\tilde{S} \in \tilde{\mathcal{M}}$ is a copy of $\tilde{S}$. As a set, we have

$$\tilde{\mathcal{E}} = \{(\tilde{S}, p) \ : \ \tilde{S} \in \tilde{\mathcal{M}} \text{ and } p \in \tilde{S}\}.$$

In the *immersive topology* on $\tilde{\mathcal{E}}$, a sequence $(\tilde{S}_n, p_n) \in \tilde{\mathcal{E}}$ converges to $(\tilde{S}, p) \in \tilde{\mathcal{E}}$ if both of the following hold:

- The sequence $\tilde{S}_n$ converges to $\tilde{S}$ in the immersive topology on $\tilde{\mathcal{M}}$.
- For one (or equivalently all) closed disk $K \subset S$ containing p and the basepoint, the immersions $\iota_n : K \rightsquigarrow \tilde{S}_n$ (which exist for n sufficiently large) satisfy $d_n\big(p_n, \iota_n(p)\big) \to 0$ as $n \to \infty$, where d_n is the metric on $\tilde{S}_n$.

(Again, see [Hoo13b] for a formal definition.)

Given topologies on $\tilde{\mathcal{M}}$ and $\tilde{\mathcal{E}}$, there is a canonical way to topologize the space $\mathcal{M}$ of all (pointed) translation surfaces. Namely given a translation surface $S \in \mathcal{M}$, we consider its universal cover $\tilde{S} \in \tilde{\mathcal{M}}$ and consider all lifts of the basepoint of S to $\tilde{S}$. So, a sequence of translation surfaces $S_n \in \mathcal{M}$ converges to $S \in \mathcal{M}$ if both of the following hold:

- The sequence of universal covers $\tilde{S}_n$ converges to $\tilde{S}$ in $\tilde{\mathcal{M}}$.
- A point $\tilde{p} \in \tilde{S}$ is a lift of the basepoint of S if and only if there is a sequence $\tilde{p}_n \in \tilde{S}_n$, with each $\tilde{p}_n$ a lift of the basepoint of S_n, so that $(\tilde{S}_n, \tilde{p}_n)$ converges to $(\tilde{S}, \tilde{p})$ in $\tilde{\mathcal{E}}$.

We also topologize the translation surface bundle $\mathcal{E}$ over $\mathcal{M}$, but we will not define it here. (See [Hoo13a] for formal definitions.)

We will now highlight some of the main results from [Hoo13a] involving these topologies.

Theorem 2. *The immersive topologies on $\tilde{\mathcal{M}}$, $\tilde{\mathcal{E}}$, $\mathcal{M}$ and $\mathcal{E}$ are second countable and Hausdorff.*

In particular, note that convergent sequences have unique limits. Furthermore, there is only one obstruction to finding a convergent subsequence of a sequence in $\mathcal{M}$:

Theorem 3. *For any $\epsilon > 0$, the set of surfaces in $\mathcal{M}$ for which the basepoint's open ϵ-neighborhood is isometric to the open ϵ-ball in the plane is compact.*

Dynamics

We would also like to utilize this topology to understand related dynamical systems. The straight line flow on $\mathcal{M}$ in the direction of θ moves the basepoint in direction θ at unit speed. Similar straight-line flows can be defined on the spaces $\tilde{\mathcal{M}}$, $\tilde{\mathcal{E}}$, and $\mathcal{E}$, and it can be shown that these flows are continuous wherever they are defined. The general linear group $\mathrm{GL}(2, \mathbb{R})$ also acts on these spaces, and it can be shown that the actions are jointly continuous. As a consequence to this, we can prove a result about how affine automorphisms behave under limits.

Theorem 4. *If S_n converges to S in $\mathcal{M}$, and there is a sequence $\phi_n : S_n \to S_n$ of affine automorphisms whose derivatives converge in $GL(2, \mathbb{R})$ and the the images under ϕ_n of the basepoints converge to some point $(S, p) \in \mathcal{E}$, then S has an affine automorphism with the limiting derivative which sends the basepoint to p.*

We have the following analog of Masur's criterion for unique ergodicity [Mas92].

Theorem 5. *Suppose $S \in \mathcal{M}$ is a translation surface of area one. If there is a sequence of times $t_n \to \infty$ and a sequence of basepoints s_n of S so that under the Teichmüller flow $g^{t_n}(S, s_n)$ converges to a unit area surface in $\mathcal{M}$, then the vertical straight line flow in the vertical direction is uniquely ergodic.*

This result can be deduced from work of Treviño [Tre14], but has not yet appeared.

References

[Hoo13a] W. Patrick Hooper, *Immersions and the space of all translation structures*, 2013, preprint, arXiv:1310.5193.

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The City College of New York, New York, NY, USA 10031

E-mail address: `whooper@ccny.cuny.edu`