Lower bounds on growth rates of periodic billiard trajectories in some irrational polygons

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Given a polygon P, how does the number of (combinatorially distinct) periodic billiard trajectories of length less than t grow?

For polygons whose angles are rational multiples of π, the answer is quite well understood. But, for irrational polygons very little is known. More details on the history of this question are given below.

We will consider a refinement of the above question. How fast *can* the number of periodic trajectories of length less than t grow in an irrational polygon P? We will provide examples of irrational polygons where this growth rate is at least $t \log^k t$ for any k.

Rather than dealing directly with irrational polygons, we choose to deal with open sets of polygons. Let $\mathcal{P}_m$ denote the space of polygons of unit area with m vertices. This space inherits a topology from the inclusion $\mathcal{P}_m \subset (\mathbb{R}^2)^m$. Number the edges of the polygons $1, \ldots, m$. Associated to a periodic billiard trajectory is a bi-infinite repeating sequence of numbered edges a billiard path hits. The *orbit-type* of a periodic billiard path is an equivalence class of such bi-infinite sequences. The equivalence class accounts for all such sequences that arise from different parameterizations of a periodic billiard path. A precise definition will be given in section 2.

Let $N(P,t)$ denote the number of orbit-types of periodic billiard trajectories of length less than t in the polygon P. For a subset $U \subset \mathcal{P}_m$, let $N(U,t)$ be the number of orbit types that appear as periodic billiard paths of length less than t in *every* polygon $P \in U$.

Theorem 1. *For each $m \geq 3$ and every $k \in \mathbb{N}$ there exist an open set*

$U \subset \mathcal{P}_m$ so that

$$\liminf_{t\to\infty} \frac{N(U,t)}{t\log^k t} > 0$$

Since open sets contain irrational polygons, we have:

Corollary 2. *For each $m \geq 3$ and each $k \in \mathbb{N}$ there is an irrational polygon P with m sides where*

$$\liminf_{t\to\infty} \frac{N(P,t)}{t\log^k t} > 0$$

Katok has shown that given any simply connected polygon P, the quantity $N(P,t)$ grows subexponentially [Kat87]. For general lower bounds in irrational polygons almost nothing is known. For instance, it is unknown if every polygon has a periodic billiard trajectory. (See [Sch06a] and [Sch06b] for some recent progress on triangles.) For right triangles, it is known that $N(P,t)$ grows at least linearly. (See figure 50 of [VGS92].) Our results hold for polygons with no restriction on the rank of the subgroup of S^1 generated by rotations by angles which appear in the polygon.

However, much stronger results are known in the case of rational polygons. Indeed, Masur showed that in a fixed rational polygon, P, the quantity $N(P,t)$ has both quadratic lower [Mas88] and upper bounds [Mas90]. Other proofs of this fact were given later by Eskin and Masur [EM01] and Vorobets [Vor05].

In the next section, we introduce some ideas coming from rational billiards. We apply these ideas to computing lower bounds for the growth rate of closed geodesics in some translation surfaces with boundary. Section two is devoted to finding open conditions which are sufficient for finding an immersion of our translation surfaces with boundary into a Euclidean cone surface. These ideas are then applied to billiards. The remaining three sections of this paper are devoted to proofs which are too long to be included in sections 1 and 2.

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1 Counting trajectories on translation surfaces with boundary

As the study of rational billiards has revealed, the natural objects in the study of billiards are translation surfaces, and the natural maps between them are affine.

A *translation surface* is a surface that can be constructed from polygonal subsets of the plane glued together by translations. These surfaces are allowed to have cone points which are integer multiples of 2π. In the study of rational billiards, translations surfaces are generally closed surfaces. However, we will be considering *translation surfaces with boundary*, translation surfaces with piecewise linear boundaries. Cone points can appear on the boundary. For cone points on the boundary, there is no restriction on the possible cone angle.

We will now give our primary example of a translation surface with boundary. Let n be an integer greater than 1 and let θ be a real number satisfying $0 < \theta < \frac{2\pi}{n}$. Then we form two $n+1$ sided polygons from the convex hulls of the points in the complex plane:

$$P_k = e^{k\theta i} \text{ and } Q_k = -e^{k\theta i} \text{ for } 0 \leq k \leq n$$

See figure 1. Glue each edge $\overline{P_k P_{k+1}}$ to $\overline{Q_{k+1} Q_k}$ by a translation. Call the resulting translation surface with boundary $S(n, \theta)$.

The boundary of $S(n, \theta)$ consists of the union of the two segments $\overline{P_0 P_n}$ and $\overline{Q_n Q_0}$. There are two cone points on the boundary consisting of identified vertices of the polygon. No cone points appear in the interior.

Now consider $SL(2, \mathbb{R})$ acting linearly on the plane. Given a translation surface S, built from polygonal pieces $\Pi_1, \ldots, \Pi_m$ of the plane as above, and an $A \in SL(2, \mathbb{R})$, we can define $A(S)$ to be the same gluing of $A(\Pi_1), \ldots, A(\Pi_m)$. This gluing makes sense, since the action of A on the plane takes parallel lines to parallel lines and preserves ratios of lengths of parallel segments. This description gives a homeomorphism from S to $A(S)$.

An affine automorphism of a translation surface is an element A so that S and $A(S)$ are isometric.

Let C_k for $k = 0 \ldots \lfloor \frac{n-1}{2} \rfloor$ be the cylinder formed by the two quadrilaterals

$$C_k = P_k P_{k+1} P_{n-k-1} P_{n-k} \ \cup \ Q_k Q_{k+1} Q_{n-k-1} Q_{n-k}$$

These are formed from the bands illustrated in figure 1.

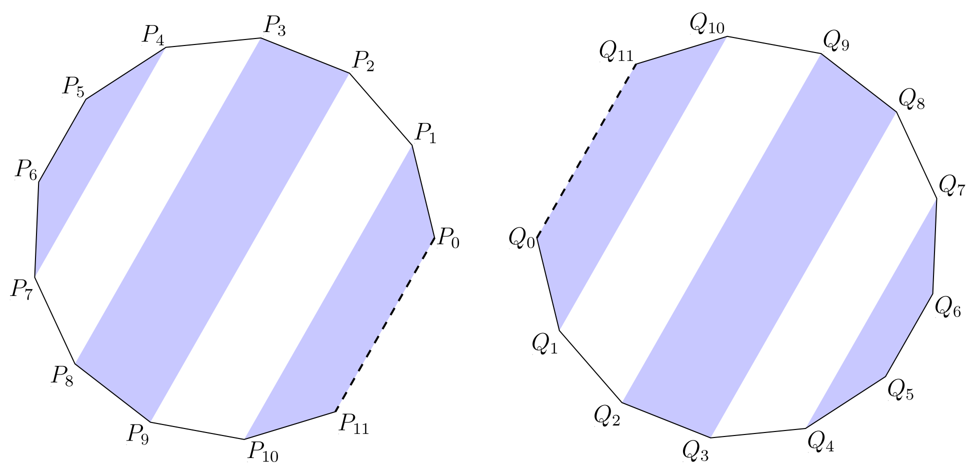

Figure 1: The surface $S(11, \theta)$ for some θ.

The fact that this surface has a parabolic automorphism is essentially the same as the proof that the surfaces Veech built from regular polygons do. [Vee89].

Proposition 3 (Automorphism)**.** *There is a parabolic automorphism of* $S(n, \theta)$ *that acts as a Dehn twist in each* C_k.

Proof. It follows from work of Veech [Vee89] that we only need prove that the moduli of the cylinders are equal. See also [MT02]. The *modulus* of a cylinder is the ratio of the cylinder's width to its circumference. Let $z = e^{\theta i}$. Then the vertices of our polygons are $P_k = z^k$ and $Q_k = -z^k$.

The length the circumference of the cylinder is the absolute value of the translational holonomy around the cylinder. This translational holonomy around C_k is given by the formula

$$\begin{aligned} l_k &= (P_{n-k} - P_k) + (Q_{k+1} - Q_{n-k-1}) \\ &= z^{n-k} - z^k + z^{n-k-1} - z^{k+1} \\ &= (z+1)(z^{n-k-1} - z^k) \end{aligned} \tag{1}$$

The width of the cylinder C_k is given by the absolute value of the quantity

$$\begin{aligned} w_k &= \tfrac{1}{2}((P_{k+1} - P_k) + (P_{n-k-1} - P_{n-k})) \\ &= \tfrac{1}{2}(z^{k+1} - z^k + z^{n-k-1} - z^{n-k}) \\ &= \tfrac{1}{2}(z-1)(z^k - z^{n-k-1}) \end{aligned} \tag{2}$$

Thus, the modulus of the cylinder C_k is given by

$$m_k = |\frac{w_k}{l_k}| = |\frac{z-1}{2(z+1)}| \tag{3}$$

The modulus is independent of k. Thus, there is a parabolic which simultaneous Dehn twists in each cylinder. $\diamondsuit$

The other fact we will need is that $S(n,\theta)$ isometrically embeds into $S(n+1,\theta)$. This embedding is shown in figure 2.

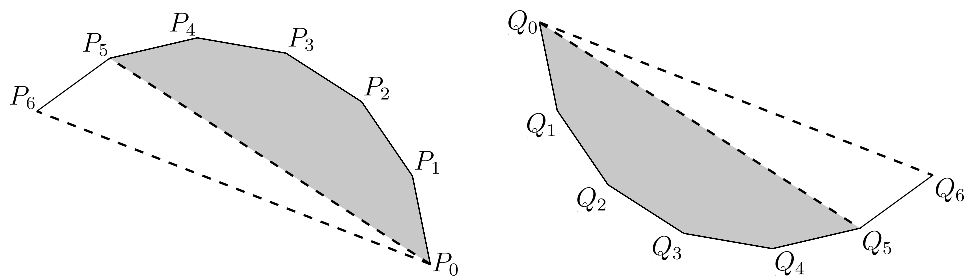

Figure 2: An isometric embedding of $S(5,\theta)$ into $S(6,\theta)$ for some θ.

We will be counting closed trajectories in the surfaces $S(n,\theta)$.

Theorem 4. *Let $N_{n,\theta}(t)$ denote the number of homotopy classes of curves in $S(n,\theta)$ that contain a closed geodesic of length less than t. Then, for $n \geq 3$,*

$$\liminf_{t\to\infty} \frac{N_{n,\theta}(t)}{t(\log t)^{n-3}} > 0$$

The genus of the surface $S(n,\theta)$ is $n-2$. If the answer to the following question is affirmative, we might be able to build billiard tables where the number of periodic trajectories grow quicker than the examples in this article.

Question 5. *Are there translation surfaces with boundary of genus $g > 0$ where $N(t)$ grows faster than $t(\log t)^{g-1}$?*

A *left Dehn twist* d in the collection of disjoint curves $\gamma_1, \ldots, \gamma_k$ acts on a homotopy class $[\alpha]$. The curve $d(\alpha)$ is homotopic to the curve which follows α until it hits one of the γ_i, then turns left and travels around γ_i for one full loop, then continues along α. At each intersection with a γ_i it does one full loop around γ_i to the left before continuing.

Let us sketch the idea of the proof of theorem 4. We will let d_n denote the affine automorphism of $S(n,\theta)$. We choose d_n so that it acts as a *left* Dehn twist in the cylinders of figure 1. We will let i_n be the inclusion $S(n,\theta) \to S(n+1,\theta)$.

Note that $S(2,\theta)$ is simply a single cylinder. We will let $\gamma_2 \subset S(2,\theta)$ be a geodesic which travels around this cylinder. Orient γ_2 so that it travels from segment $\overline{P_0P_1}$ to $\overline{P_1P_2}$ through the left polygon of figure 1 and then from $\overline{Q_2Q_1}$ to $\overline{Q_1Q_0}$ in the right polygon. We will inductively define a collection of geodesic curves Γ_n on $S(n,\theta)$. Let $\Gamma_2 = \{\gamma_2\}$. Let

$$\Gamma_{n+1} = \{d_{n+1}^k \circ i_{n+1}(\gamma) \text{ where } k \geq 0 \text{ and } \gamma \in \Gamma_n\}. \tag{4}$$

Note that we only consider non-negative powers of our affine transformation d_{n+1}. Thus, we leave out some obvious closed geodesics on $S(n,\theta)$. We do not believe this affects our lower bound.

Then our theorem 4 follows from the following two lemmas.

Lemma 6 (Distinctness)**.** *No two curves in Γ_n are homotopic.*

Lemma 7 (Counting)**.** *Let $N_n(t)$ denote the number of elements of Γ_n whose lengths are less than t. Then for $n \geq 3$,*

$$\liminf_{t\to\infty} \frac{N_n(t)}{t(\log t)^{n-3}} > 0$$

We prove these lemmas in sections 3 and 4.

2 $S(n,\theta)$ appears in many polygons

A *scaled isometric immersion* from a cone surface A to a cone surface B, $f : A \to B$, is an immersion which distorts distances by a constant. That is, there is a constant $c > 0$ so that for all $a_1, a_2 \in A$

$$dist(f(a_1), f(a_2)) = c\ dist(a_1, a_2)$$

We will next describe an *open condition* that implies the existence of scaled isometric immersions of $S(n,\theta)$ into a Euclidean cone surface X. The existence of this immersion will imply that the growth rate of closed geodesics has a lower bound which similar to the lower bound for $S(n, \theta)$.

Before we describe this open condition, let us introduce some notation. X will be an oriented Euclidean cone surface. Assume X has a cone point P with cone angle $\varphi < \pi$. Let $\theta = \pi - \varphi$. Let σ be a saddle connection joining the cone point P to itself. We will use l to denote the length of σ.

We will say that a map $g : A \to B$ is $k : 1$ ("k to 1") if each $b \in B$ has at most k preimages in A.

Lemma 8 (Immersion)**.** *Assume X, P, φ, θ, σ, and l are as above. Let h be the constant depending on the integer $n \geq 2$*

$$h = \frac{n^2 l \theta}{8}.$$

If the h-neighborhood of σ, $\{x \in X \text{ so that } dist(\sigma, x) < h\}$, contains no cone points of X other than P, then there is a scaled isometric immersion of $S(n, \theta)$ into X branched over P which is at most $2n + 2 : 1$.

We will prove this lemma in section 5.

For a Euclidean cone surface X let $N(X, t)$ denote the number of homotopy classes of curves on X which contain closed geodesics of length less than t.

Proposition 9. *Suppose there is a scaled isometric immersion f of $S(n, \theta)$ into a Euclidean cone surface X, which is at most $k : 1$ and is branched over the cone points. Then,*

$$\liminf_{t \to \infty} \frac{N(X, t)}{t(\log t)^{n-3}} > 0.$$

Proof. Let $c > 0$ be the constant coming from the scaled isometric immersion. Thus, if $\gamma \subset S(n, \theta)$ is a closed geodesic of length t, $f(\gamma)$ is a closed geodesic of length ct. Since f is at most $k : 1$, it maps at most k homotopy classes containing geodesics to a homotopy class containing a geodesic. Therefore

$$N(X, t) \geq \frac{N_n(t/c)}{k}.$$

Here $N_n(t/c)$ denotes the number of closed geodesics of length less than t/c in $S(n, \theta)$. By theorem 4, it follows that

$$\liminf_{t \to \infty} \frac{N(X, t)}{t(\log t)^{n-3}} \geq \lim_{t \to \infty} \frac{N_n(t/c)}{kt(\log t)^{n-3}} = \liminf_{t \to \infty} \frac{N_n(t)}{kct(\log t + \log c)^{n-3}} > 0.$$

$\diamondsuit$

By combining the immersion lemma with the above proposition, we see that the conditions of the immersion lemma are enough to guarantee that the growth rate of closed geodesics on a Euclidean cone surface are at least $t(\log t)^{n-3}$.

It remains to verify the criteria of the lemma on some polygonal billiard tables. Let P be a polygon with m sides. As in the introduction, number the edges of P from $1 \ldots m$. A typical billiard trajectory hits some infinite sequence of edges in both forward and backward time. Thus, associated to a typical trajectory is a bi-infinite sequence $\langle \alpha_i \rangle_{i \in \mathbb{Z}}$ with each $\alpha_i \in \{1, \ldots, m\}$. The forward sequence $\langle \alpha_i \rangle_{i \geq 0}$ represents the sequence of numbers marking the edges hit by the trajectory in forward time, and $\langle \alpha_i \rangle_{i<0}$ represents the numbers marking edges hit in backward time. In particular, if the billiard trajectory is periodic, then this bi-infinite sequence is also periodic. The *orbit-type* of a billiard path is an equivalence class of such bi-infinite sequences. The equivalence is up to shifts and reflections. That is, $\langle \alpha_i \rangle \sim \langle \beta_i \rangle$ if there is an $n \in \mathbb{Z}$ so that $\alpha_i = \beta_{i+n}$ for all $i \in \mathbb{Z}$ or if there is an $n \in \mathbb{Z}$ so that $\alpha_i = \beta_{n-i}$ for all $i \in \mathbb{Z}$. This accounts for the ambiguity arising from different parameterizations of a billiard trajectory yielding different bi-infinite sequences. The passage from bi-infinite sequence to orbit-type is analogous to the passage from an element of the fundamental group of a surface to a homotopy (or conjugacy) class.

For a polygon P, we let $N(P, t)$ denote the number of orbit-types for which there is a periodic billiard path of length less than t of that orbit-type.

Let $\mathcal{D}P$ denote the Euclidean cone surface obtained by doubling P across its boundary. This surface is topologically a sphere with a flat metric and cone singularities corresponding to the vertices of P. Picture a pillow case made from two copies of P. There is a natural $2:1$ folding map $f : \mathcal{D}P \to P$ which sends closed geodesics on $\mathcal{D}P$ to billiard paths on P. Further, this map sends closed geodesics in the same homotopy class to billiard paths with the same orbit-type. This induced map from closed geodesics to billiard paths is $2:1$ in the sense that for each orbit-type of a periodic billiard path on P, there are at most two homotopy classes of curves in $\mathcal{D}P$ which contain closed geodesics which are mapped under f to billiard paths with that orbit-type.

By an argument similar to that used in proposition 9, if $\mathcal{D}P$ satisfies the conditions of lemma 8, then

$$\liminf_{t\to\infty} \frac{N(P,t)}{t(\log t)^{n-3}} > 0.$$

Proof of Theorem 1. We will prove this case for triangles. The extension to polygons with more than three sides will be described at the end of the proof.

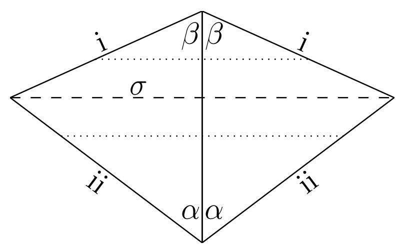

Figure 3: This figure shows the double $\mathcal{D}T$ of an acute triangle T with angles α and β. The dotted lines represent the boundary of an h neighborhood of the saddle connection σ.

Consider the acute triangle T with angles $(\alpha, \beta, \pi - \alpha - \beta)$. We will assume that $\alpha < \beta$. Also consider the shortest saddle connection σ joining the cone point of angle $\varphi = 2(\pi - \alpha - \beta)$ to itself. See figure 3. Let l be the length of σ, and $\theta = \pi - \varphi = 2\alpha + 2\beta - \pi$ as in the statement of the immersion lemma. We let

$$h = \frac{n^2 l\theta}{8} = \frac{n^2 l(2\alpha + 2\beta - \pi)}{8}.$$

as in the immersion lemma. Note that the h neighborhood of σ contains no other cone points so long as $h < \frac{l}{2\tan\beta}$. Thus the conditions of the immersion lemma are satisfied if

$$\frac{n^2 l(2\alpha + 2\beta - \pi)}{8} < \frac{l}{2\tan\beta}.$$

This is equivalent to

$$n^2(2\alpha + 2\beta - \pi)\tan\beta < 4$$

We can see that this condition is true on an open set for all n. Fix $\frac{\pi}{4} < \beta < \frac{\pi}{2}$. We can choose $\alpha = \frac{\pi}{2} - \beta + \epsilon$ to make $2\alpha + 2\beta - \pi = 2\epsilon$ arbitrarily small while still maintaining $\alpha < \beta$.

By the argument before the theorem, we can use proposition 9 to show that

$$\liminf_{t\to\infty} \frac{N(T,t)}{t\log^{n-3} t} > 0$$

Now we will show that this argument extends to polygons with more sides. If we deform the double $\mathcal{D}T$ outside the h-neighborhood of σ, it will still satisfy lemma 8. In particular, we can deform T outside the h-neighborhood of the image of σ under the folding map $\mathcal{D}T \to T$. (This image is an altitude of the triangle T.) $\diamondsuit$

3 Proof of the distinctness lemma

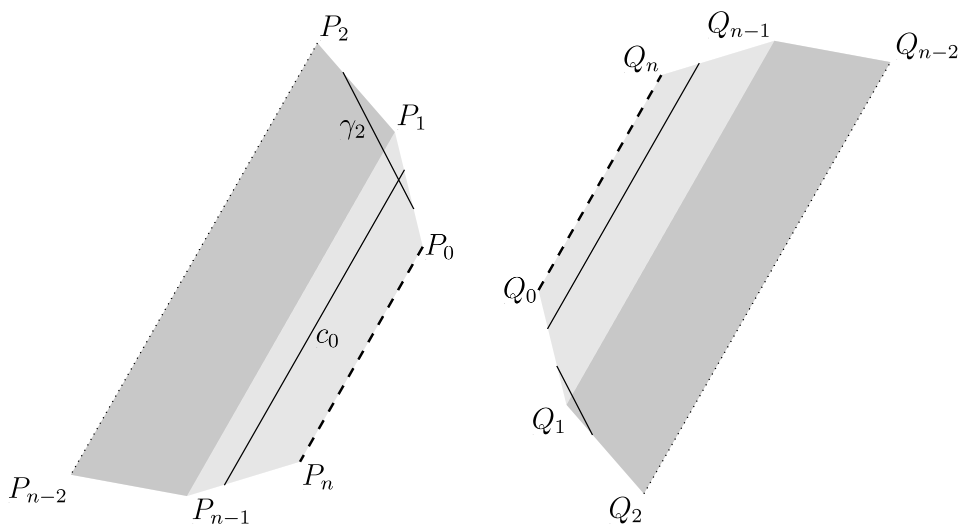

Figure 4: The portion of the surface $S(n,\theta)$ relevant to the proof of the distinctness lemma.

Recall that we defined Γ_n inductively via formula 4. Elements of Γ_{n+1} are constructed by choosing an element $\gamma \in \Gamma_n$ and left Dehn twisting. That is, $d_{n+1}^k \circ i_{n+1}\gamma \in \Gamma_{n+1}$ for $k \geq 0$.

The proof of the distinctness lemma (lemma 6) essentially follows from the following proposition.

Proposition 10. *For all $\gamma \in \Gamma_n$, the algebraic intersection between $\overrightarrow{P_0P_1} = \overrightarrow{Q_1Q_0}$ and γ, denoted $\#(\overrightarrow{P_0P_1}, \gamma)$, is positive. In fact, if $\eta = d_{n+1}^k \circ i_{n+1}\gamma \in \Gamma_{n+1}$ for some $k \geq 0$ then*

$$\#\big(\overrightarrow{P_0P_1}, \eta) = (k+1)\#(\overrightarrow{P_0P_1}, \gamma) > 0.$$

Proof. This statement is clearly true for $n = 2$, since $\Gamma_2 = \{\gamma_2\}$. We will apply induction. Let $\gamma \in \Gamma_n$ and suppose $\#(\gamma, \overrightarrow{P_0P_1}) > 0$. We will show that the curve $d_{n+1}^k \circ i_{n+1}(\gamma) \in \Gamma_{n+1}$ for $k \geq 0$ satisfies the proposition as well. Consider the curve c_0 which travels around the outermost cylinder of $S(n+1, \theta)$. Orient c_0 so that it travels from $\overline{P_0P_1}$ to $\overline{P_nP_{n+1}}$ and from $\overline{Q_{n+1}Q_n}$ to $\overline{Q_1Q_0}$. We claim that

$$\#\big(i_{n+1}(\gamma), c_0\big) = \#(\overrightarrow{P_0P_1}, \gamma). \tag{5}$$

This is because c_0 is homotopic to the polygonal path

$$\overrightarrow{P_1P_0} \cup \overrightarrow{P_0P_n} \cup \overrightarrow{Q_{n+1}Q_n} \cup \overrightarrow{Q_nQ_0}$$

(Note that the edges $\overrightarrow{P_0P_1}$ and $\overrightarrow{P_nP_{n+1}}$ are identified as are $\overrightarrow{P_nP_{n+1}}$ and $\overrightarrow{Q_{n+1}Q_n}$.) The arcs other than $\overrightarrow{P_1P_0}$ in this polygon consist of boundary components or are outside $i_{n+1}(S(n, \theta))$, so that equation 5 must be true. Thus $\#(i_{n+1}(\gamma), c_0)$ is positive. Then we apply the formula for the action of a left Dehn twist in the multicurve $\{c_0, c_1, c_2, \ldots\}$,

$$[\![d_{n+1}^k \circ i_{n+1}(\gamma)]\!] = [\![i_{n+1}(\gamma)]\!] + k\Big(\sum_{j=0\ldots} \#\big(i_{n+1}(\gamma), c_j\big)[\![c_j]\!]\Big). \tag{6}$$

The other curves $c_1, c_2, \ldots$ are curves which travel around the remaining cylinders of the decomposition. These curves do not intersect $\overrightarrow{P_0P_1}$. On the other hand, $i(\overrightarrow{P_0P_1}, c_0) = 1$. We will now use the fact that intersection number is a homological invariant. Equation 6 entails that

$$\begin{aligned} \#\big(\overrightarrow{P_0P_1}, d_{n+1}^k \circ i_{n+1}(\gamma)\big) &= \#\big(\overrightarrow{P_0P_1}, i_{n+1}(\gamma)\big) + k\#\big(i_{n+1}(\gamma), c_0\big)\#(\overrightarrow{P_0P_1}, c_0) \\ &= (k+1)\#(\overrightarrow{P_0P_1}, \gamma) > 0. \end{aligned} \tag{7}$$

$\diamondsuit$

Proof of the distinctness lemma. Again we proceed by induction. Since Γ_2 consists of only one curve, it is obviously true for $n = 2$. Assume that no pair of distinct curves in Γ_n are homotopic. Now suppose the two curves $\eta = d^k_{n+1} \circ i_{n+1}(\gamma)$ and $\eta' = d^{k'}_{n+1} \circ i_{n+1}(\gamma')$ in Γ_{n+1} are homotopic. We can assume that $\gamma' \neq \gamma$, for if $\gamma' = \gamma$ and $k \neq k'$, the intersection formula given in proposition 10 distinguishes η from η'. Thus we may assume γ distinct from γ' in Γ_n. We need to show that $k = k'$. This is sufficient, for if $k = k'$ then we can apply d^{-k}_{n+1} to demonstrate that $i_{n+1}(\gamma)$ is homotopic to $i_{n+1}(\gamma')$. This implies that γ is homotopic to γ' which is impossible by assumption. Thus, it is sufficient to recover k from the homotopy class of $d^k_{n+1} \circ i_{n+1}(\gamma)$.

We know that $\#(\overrightarrow{P_{n+1}P_n}, c_0) = 1$ while $\#(\overrightarrow{P_{n+1}P_n}, c_j) = 0$ for $j > 0$. Now let $\gamma \in \Gamma_n$. Because $\overrightarrow{P_{n+1}P_n}$ is disjoint from $i_{n+1}\big(S(n,\theta)\big)$, equation 6 tells us that

$$\begin{aligned} \#\big(\overrightarrow{P_{n+1}P_n}, d^k_{n+1} \circ i_{n+1}(\gamma)\big) &= k\#\big(i_{n+1}(\gamma), c_0\big)\#(\overrightarrow{P_{n+1}P_n}, c_0) \\ &= k\#\big(i_{n+1}(\gamma), c_0\big) \\ &= k\#\big(\overrightarrow{P_0P_1}, \gamma\big) \end{aligned} \tag{8}$$

Compare this to equation in proposition 10, which showed that

$$\#\big(\overrightarrow{P_0P_1}, d^k_{n+1} \circ i_{n+1}(\gamma)\big) = (k+1)\#(\overrightarrow{P_0P_1}, \gamma) \tag{9}$$

We will now determine k from topological information about the curve $\eta = d^k_{n+1} \circ i_{n+1}(\gamma)$. Let $a = \#(\overrightarrow{P_0P_1}, \eta)$ and $b = \#(\overrightarrow{P_{n+1}P_n}, \eta)$. By equations 8 and 9,

$$b - a = \#(\overrightarrow{P_0P_1}, \gamma)$$

which is not zero by proposition 10. Further, we see that

$$\frac{b}{b-a} = k$$

We have determined k from topological information about η as promised. $\diamondsuit$

4 Proof of the counting lemma

This section contains the proof of lemma 7.

Proof of the counting lemma. Let $\ell(\gamma)$ denote the length of the geodesic curve γ. First we will demonstrate that there is an $M > 0$ depending only on n so that

$$\ell(d_{n+1}^k(\gamma)) \leq (kM+1)\ell(\gamma) \tag{10}$$

for all geodesic curves γ on $S(n+1,\theta)$.

Recall that the formula the action of the left Dehn twist d_{n+1} in the multicurve $\gamma_1, \ldots, \gamma_m$ on homology is given by

$$[\![d_{n+1}^k(\gamma)]\!] = [\![\gamma]\!] + k \sum_{i=1}^{m} i(\gamma, \gamma_i)[\![\gamma_i]\!], \tag{11}$$

where $i(\gamma, \gamma_i)$ denotes the algebraic intersection number. The length of a geodesic γ is the translation distance of the holonomy around γ, which is a homology invariant. Let $hol([\![\gamma]\!])$ denote the translation vector determined by the holonomy around γ, and let $|hol([\![\gamma]\!])|$ denote the length of this translation vector. Applying the triangle inequality to equation 11 yields

$$\ell(d_{n+1}^k(\gamma)) \leq \ell(\gamma) + k\Big|hol\Big(\sum_{i=1}^{m} i(\gamma, \gamma_i)[\![\gamma_i]\!]\Big)\Big| \tag{12}$$

Our d_{n+1} is induced by an affine transformation of the plane A. We can take the action of A on the plane to be the linear action of a 2×2 matrix. Consider the new linear transformation of the plane $B(x) = A(x) - x$ for $x \in \mathbb{R}^2$. Then $|B(x)|/|x|$ is an invariant of a vector's direction ($|B(kx)|/|kx| = |B(x)|/|x|$). Further, there is an $M > 0$ so that the quantity $|B(x)|/|x| < M$ for all x. By equation 11,

$$hol\Big(\sum_{i=1}^{m} i(\gamma, \gamma_i)[\![\gamma_i]\!]\Big) = hol([\![d_{n+1}(\gamma)]\!]) - hol([\![\gamma]\!]) = B(hol([\![\gamma]\!])) \tag{13}$$

And $|B(hol([\![\gamma]\!]))| \leq M|hol([\![\gamma]\!])| = M\ell(\gamma)$. Thus, equation 13 together with equation 12 implies our claim, equation 10.

Using equation 10, we will prove the lemma by induction. Γ_2 consists of exactly one curve, γ_2. Let $l = \ell(\gamma_2)$. Then, $\Gamma_3 = \{i_3(\gamma_2), d_3 \circ i_3(\gamma_2), d_3^2 \circ i_3(\gamma_2), \ldots\}$, so by equation 10,

$$N_3(t) \geq \lfloor \frac{t-l}{Ml} \rfloor + 1 \geq \frac{t-l}{Ml}$$

So indeed
$$\liminf_{t\to\infty} \frac{N_3(t)}{t} \geq \frac{1}{Ml} > 0.$$

Now suppose we know there are constants $c > 0$ and $T > 0$ so that for $t > T$
$$N_n(t) \geq ct(\log t)^{n-3}. \tag{14}$$
This is equivalent to the statement of the lemma. We wish to prove a statement similar to equation 14 for $N_{n+1}(t)$. By equation 10, we know that
$$N_{n+1}(t) \geq \sum_{k=0}^{\infty} N_n(\frac{t}{kM+1}). \tag{15}$$
Here, $N_n(\frac{t}{kM+1})$ is greater than or equal to the number of elements γ of Γ_n so that $\ell(d_{n+1}^k \circ i_{n+1}(\gamma)) < t$. By equation 14, for all $t > (kM+1)T$ (or equivalently $k < \frac{t-T}{MT}$), we have
$$N_n(\frac{t}{kM+1}) \geq \frac{ct}{kM+1}(\log\frac{t}{kM+1})^{n-3} \tag{16}$$
It is a classical exercise in number theory to see that equations 15 and 16 imply the lemma. See chapter 18 of [HW79], for instance. However, in the interest of completeness, we provide the short proof below.

By equations 15 and 16, we see
$$N_{n+1}(t) \geq \sum_{k=0}^{\frac{t-T}{MT}} N_n(\frac{t}{kM+1}) \geq \sum_{k=0}^{\frac{t-T}{MT}} \frac{ct}{kM+1}(\log\frac{t}{kM+1})^{n-3} \tag{17}$$
Since the addends are decreasing with k we can replace this sum with an integral. Assuming $t > T + MT$,
$$\begin{aligned}\sum_{k=0}^{\frac{t-T}{MT}} \frac{ct}{kM+1}(\log\frac{t}{kM+1})^{n-3} &\geq \int_0^{\frac{t-T}{MT}-1} \frac{ct}{kM+1}(\log\frac{t}{kM+1})^{n-3}\mathrm{d}k \\ &= F(\frac{t-T}{MT}-1) - F(0),\end{aligned}$$
where F is the antiderivative
$$F(k) = \frac{-ct}{(n-2)M}(\log\frac{t}{kM+1})^{n-2}.$$

Then

$$F(0) = \frac{-ct}{(n-2)M}(\log t)^{n-2}$$

and

$$F(\frac{t-T}{MT}-1) = \frac{-ct}{(n-2)M}(\log \frac{Tt}{t-T-MT+1})^{n-2}$$

There is a positive number $T' > T$ so that for $t > T'$,

$$\frac{Tt}{t-T-MT+1} < 2T$$

Then for $t > \max\{T+MT, T'\}$

$$F(\frac{t-T}{MT}-1) \geq \frac{-ct}{(n-2)M}\log 2T \geq -c_2 t$$

For some $c_2 > 0$. That is, $F(\frac{t-T}{MT}-1)$ is no worse than asymptotically linear. Thus for $t > \max\{T+MT, T'\}$,

$$\begin{aligned} N_{n+1}(t) &\geq F(\frac{t-T}{MT}-1) - F(0) \\ &= -c_2 t + \frac{ct}{(n-2)M}(\log t)^{n-2}. \end{aligned}$$

It follows that

$$\liminf_{t\to\infty} \frac{N_{n+1}(t)}{t(\log t)^{n-2}} > 0,$$

which completes the inductive step and proves the lemma. $\diamondsuit$

5 Proof of the immersion lemma

As in the statement of lemma 8, we will suppose we have an oriented Euclidean cone surface X. Suppose $P \in X$ is a cone point with cone angle $\varphi < \pi$, and that σ is a saddle connection of length l joining P to itself. We let $\theta = \pi - \varphi$. We assume that the h-neighborhood of σ contains no cone points other than P and that

$$h \geq \frac{n^2 l\theta}{8}$$

We will provide a scaled isometric immersion of $S(n,\theta)$ into X branched over P which is at most $2n+2:1$.

Proof of the immersion lemma: The saddle connection σ cuts P into two angles α and β satisfying $\alpha + \beta = \varphi$. The fact that there are no other cone singularities with distance h from σ tells us that we can find a subsurface $X_1 \subset X$ as in figure 5.

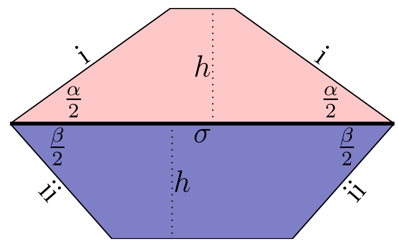

Figure 5: The subsurface $X_1 \subset X$. Roman numerals indicate gluings. X_1 is the h-neighborhood of σ and hence is topologically an annulus.

In this proof, we will repeatedly take branched covers of X_1 until we find a subsurface Y we like better. The surface Y we find is shown in figure 7. Figure 6 shows a portion of the cover of X which contains Y. The next two paragraphs detail the construction of this cover.

Let X_2 be the double branched cover of X_1 over the cone point P. X_2 is topologically a sphere with three holes. Let σ_2^1 and σ_2^2 be the two lifts of $\sigma \subset X_1$ to X_2. Consider the cover X_3 of X_2 corresponding to the subgroup of $\pi_1(X_2)$ generated by $[\sigma_2^2]^{-1}[\sigma_2^1]$. (Here σ_2^* are saddle connections connecting the cone point to itself, and hence each forms a loop whose equivalence class in $\pi_1(X_2)$ is denoted $[\sigma']$.)

Choose a lift of the path in X_2 which first travels along σ_2^1 and then backward along σ_2^2. This lift travels from one cone point along a lift σ^1 of σ_2^1 to a second cone point and then travels backward along a lift σ^2 of σ_2^2 back to the first cone point. The cone points each have cone angle 2φ and the angle made between σ^1 and σ^2 is φ at each cone point. The fact that these angles are all φ is the point of this arcane construction. We will let $Y \subset X_3$ be the h-neighborhood of $\sigma^1 \cup \sigma^2$. Because Y was built as a subsurface of a $4:1$ covering of $X_1 \subset X$, we have a branched immersion of Y into X which is at most $4:1$. An alternate construction of Y is shown in figure 6.

The surface Y can be built from four isometric isosceles trapezoids. Another picture of Y is shown in figure 7. $Y \smallsetminus (\sigma^1 \cup \sigma^2)$ consists of two annuli which are each built out of two trapezoids. Develop a chain of n copies of the trapezoid from the top annulus, and do the same for the bottom annulus.

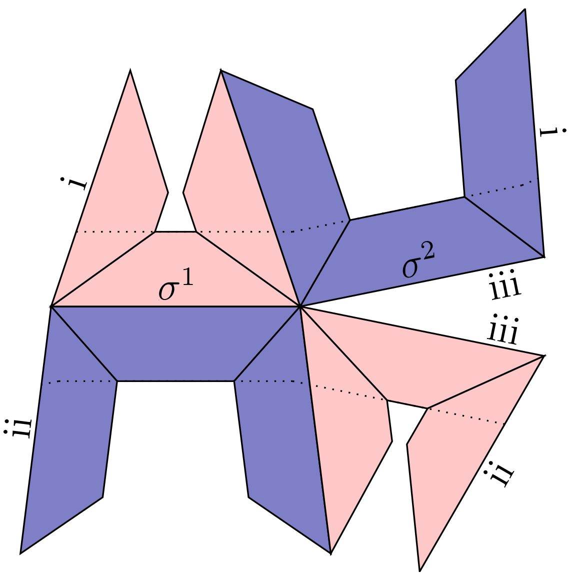

Figure 6: The whole surface pictured immerses into X_1 by a map branched over the cone points. The surface Y is the subsurface bounded by dotted lines. Topologically, Y is just an annulus.

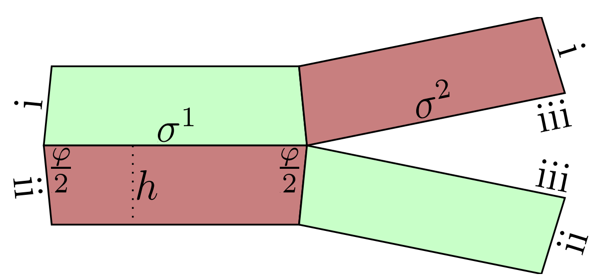

Figure 7: The surface Y. The surface can be built from four isometric isosceles trapezoids.

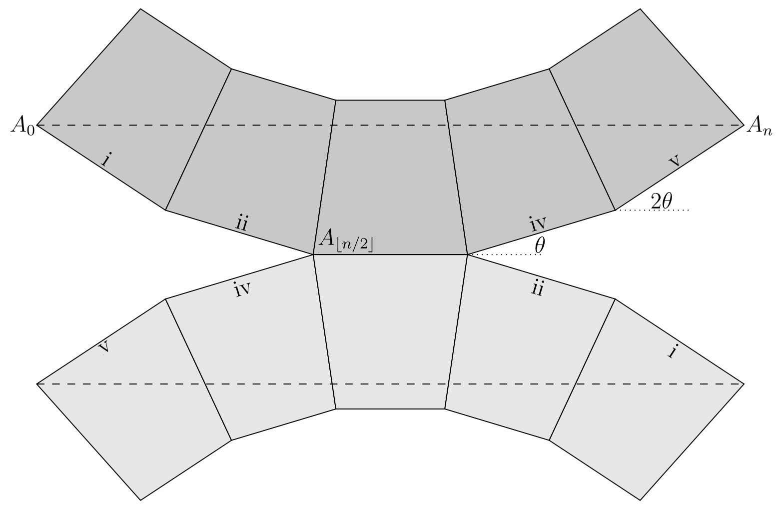

Figure 8: A scaled isometric embedding of $S(5, \theta)$ into $\widetilde{Y}$ for some θ.

See figure 8. We reglue the trapezoids along lifts of σ^1 and σ^2 by translations. The resulting surface $\widetilde{Y}$ supports a $\lceil \frac{n}{2} \rceil : 1$ branched immersion into Y. By composition, $\widetilde{Y}$ also supports a branched immersion into X which is at most $2n+1 : 2$.

Figure 8 shows an example when $S(n, \theta)$ scaled isometrically embeds into $\widetilde{Y}$. The image of $S(n, \theta)$ is bounded by the dotted lines. This embedding will always exist if $\overline{A_0 A_n}$ is contained inside $\widetilde{Y}$.

If we fix $l > 0$ and $h > 0$, for small enough θ, the segment $\overline{A_0 A_n}$ will be contained inside $\widetilde{Y}$. Let $D(\theta, l)$ denote the greatest distance between $\overline{A_0 A_n}$ and

$$\bigcup_{i=1}^{n} \overline{A_{i-1} A_i}.$$

We compute that $D(\theta, l)$ is given by the following formula

$$D(\theta, l) = \begin{cases} l \sum_{i=1}^{\frac{n-1}{2}} \sin(i\theta) & \text{if } n \text{ is odd} \\ \frac{l}{2} \sum_{i=1}^{\frac{n}{2}} \sin(i\theta) + \sin\big((i-1)\theta\big) & \text{if } n \text{ is even} \end{cases}$$

Blind substitution of α for $\sin \alpha$ everywhere reveals that $D(\theta, l) < \frac{n^2 l \theta}{8}$ in either case. For $\overline{A_0 A_n}$ to be contained in $\widetilde{Y}$, it is sufficient that $D(\theta, l) < h$.

This implies the theorem. ◇

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