Stable periodic billiard paths in obtuse isosceles triangles

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Can you place a small billiard ball on a frictionless triangular pool table and hit it so that it comes back to its its original location traveling the same direction? This is a long standing open question, but the list of triangles where it is known that you can do this is growing. This paper adds some almost isosceles triangles to this list.

Mark the edges of a triangle T by the numbers 1, 2 and 3. The *symbolic dynamics* s_γ of a periodic billiard path γ is the bi-infinite sequence of edges the billiard path hits, which we interpret as a bi-infinite sequence in $\{1,2,3\}$. A periodic billiard path γ in T is called *stable* if there is an open set of triangles U containing T so that for each $T' \in U$ there is a periodic billiard path γ' in T' with $s_\gamma = s_{\gamma'}$.

We define T_x to be the obtuse isosceles triangle with two acute angles with measure $\alpha = \frac{\pi}{2x}$ and one obtuse angle with measure $(1-\frac{1}{x})\pi$. The purpose of this paper is to prove the following:

Theorem 1 *For every $x > 2$ with $x \notin \mathbb{Z}$ there is a stable periodic billiard path in T_x.*

A *period* of a periodic billiard path is a sub-arc which starts and ends with the billiard ball traveling in the same direction at the same point on the interior of the triangle. A *symbolic period* is a finite sequence of edges hit in a period, that is a finite word in the letters $\{1,2,3\}$.

The theorem is proved by studying a two parameter family of billiard paths $Y_{n,m}$. We will mark the long side of the obtuse isosceles triangle 3 and use 1 and 2 for the shorter sides. We define the words

$$X_n = (31)^{n-1}(32)^{n-1} \tag{1}$$

We will show the words X_n are symbolic periods for unstable periodic billiard paths in isosceles triangles. We use these words to build a 2-parameter list of new words:

$$Y_{n,m} = 1(X_n)^m 32 \tag{2}$$

Theorem 1 is in fact a corollary of the result:

Theorem 2 *For every $x > 2$ with $x \notin \mathbb{Z}$, let $n \in \mathbb{Z}$ with $n < x < n + 1$. Then, there exists an $m \in \mathbb{Z}$ and a stable periodic billiard path in T_x with symbolic period $Y_{n,m}$.*

1 Stability

First we will prove that if in fact we can find a periodic billiard path with a symbolic period given by $Y_{n,m}$, then it must be stable. This follows from a well known result:

Lemma 3 (Odd periods are stable) *If γ is a periodic billiard path in the triangle T which hits an odd number of edges in its period, then γ is stable.*

Proof: The trick to the proof is look at what we will call the *unfolding* of a periodic billiard path. Picture our triangle in the plane. Each time the billiard ball hits the edge of our triangle, reflect the triangle in that edge and continue the path in that new triangle. The resulting path is a line in an bi-infinite chain of triangles $U(T, s_\gamma)$ as in figure 1.

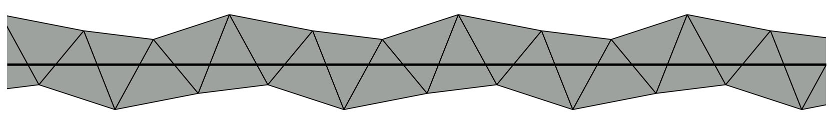

Figure 1: The unfolding for the Fagnano curve, which has symbolic period 123.

If the periodic billiard path has an odd period then $U(T, s_\gamma)$ is invariant under a glide reflection G_T. The line is the axis of this glide reflection. Generic orientation reversing isometries of the plane are glide reflections. Deform the triangle T to a triangle T'. If T' is close enough to T, then the unfolding of $U(T', s_\gamma)$ is also invariant under a glide reflection $G_{T'}$. If the axis of $G_{T'}$ stays within the unfolding and avoids the vertices of the triangles, then we can fold up the unfolding and produce a periodic billiard path in T' with symbolic dynamics s_γ. Rotate the unfolding so that it is horizontal. We must check that the vertices of $U(T, s_\gamma)$ which lie above the axis of G_T must still lie above the axis of $G_{T'}$ in the unfolding $U(T', s_\gamma)$. By the invariance of the unfolding $U(T', s_\gamma)$ under $G_{T'}$, we only need to check finitely many vertices. The distances from the vertices to the axis vary smoothly with the deformation of the triangle. So this is an open condition on T'. Therefore, there is an open set of triangles containing T with periodic billiard paths with symbolic dynamics s_γ. $\diamondsuit$

2 The unstable family X_n

We need to show that there are periodic billiard paths in these obtuse triangles with combinatorics given by X_n. The fact that these paths are unstable is irrelevant to our arguments.

Proposition 4 *For every $x > n - 1 \in \mathbb{Z}$ the triangle T_x has a periodic billiard path with symbolic period X_n.*

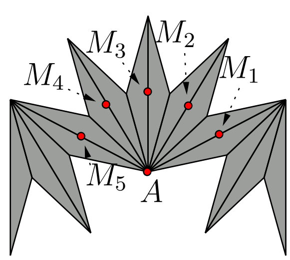

Figure 2: An unfolding for the word X_5. One period is shown.

Proof: Let M be the midpoint of the longest side of T_x. By symmetry any billiard path which passes through M twice must close up. As shown in figure 2, the measure of angle $M_1AM_n = 2(n-1)\alpha = \frac{(n-1)\pi}{x}$ is less than π precisely when $x > n-1$. In this case the polygon $AM_1M_2 \dots M_n$ is a convex subset of the unfolding. Therefore, the line segment $\overline{M_1 M_n}$ is contained in the unfolding. It is easy to verify that the symbolic period of the resulting billiard path is X_n. $\diamondsuit$

If we unfold a periodic billiard in a triangle T_x, there is a maximal strip of parallel lines contained in the unfolding. Each parallel line folds up to a periodic billiard path with the same symbolic period. The *leading vertices* of an unfolding are the vertices which are contained in the boundary of this maximal strip.

Proposition 5 *For every $x > n \in \mathbb{Z}$ the leading vertices of the billiard path with combinatorics X_n are the obtuse vertices of the rhombus containing M_1 and M_n.*

Proof: Note that the unfolding of the word X_n in an isosceles triangle is invariant under a glide reflection which sends M_1 to M_n. Also, a rotation by 180 degrees about M_1 preserves the unfolding. We will slowly go through the vertices of the unfolding (modulo these automorphisms) to show that they are not leaders. It suffices to show that the remaining vertices are further from the axis of the glide reflection. We label the claimed bottom leaders L_1 and L_n and the claimed top vertices L'_1 and L'_n. Note that there are automorphisms of the unfolding which take L_1 to each of the other claimed leaders. Therefore, the distance of each to the axis of the glide reflection is the same.

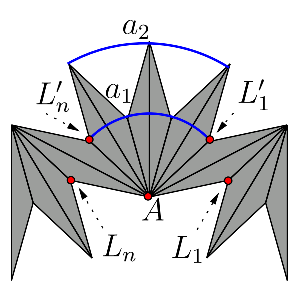

Figure 3: The point A and the points on the arcs a_1 and a_2 can not be leaders.

First we will check that the vertex labeled A is not a leader in figure 3. The measure of angle $L_1 A L_n = 2n\alpha = \frac{n\pi}{x}$, which is less than π whenever

$n < x$. When $n < x$, the point A lies further below the line than L_0 and L_n, so A cannot be a leader.

The remaining vertices we need to consider lie on two arcs, a_1 and a_2 of figure 3. The arcs are pieces of circles centered at A. The angle associated to the arcs is always less than π when $n < x$. For each i, the convex hull of L_1', L_n', and arc a_i contains the segment $\overline{L_1' L_n'}$ as a boundary component. Therefore all other points on the arc are further from the axis of the glide reflection and cannot be leaders. $\diamondsuit$

We will always normalize lengths so that the short side of the triangle T_x has length 1. We can compute the length translated by the glide reflection preserving the unfolding of the word X_n explicitly (this is the distance between M_1 and M_n in figure 2)

$$l = \sin((n-2)\alpha) + \sin(n\alpha) \tag{3}$$

We will need this in the next section.

3 The stable family $Y_{n,m}$

The word $Y_{n,m}$ has odd length. This tells us that any periodic billiard path with combinatorics $Y_{n,m}$ is stable. Also for odd words, it is easy to compute the direction of the translational holonomy.

Proposition 6 *The translation vector of the holonomy of the unfolding of an isosceles triangle by the word $Y_{n,m}$ is parallel to the long side of T_x in the first triangle of the unfolding. See figures 4 and 5.*

Proof: We will prove that there is reflective symmetry of the unfolding which preserves the first triangle and swaps edges marked 1 with edges marked 2. Such a symmetry of the unfolding induces a symmetry of the symbolic dynamics. The word $Y_{n,m}$ should be invariant under the “symmetry” which reverses order and swaps the letter 2 with the letter 1.

Recall $Y_{n,m} = 1((31)^{n-1}(32)^{n-1})^m 32$. Let $W_{n,m}$ be the word $Y_{n,m}$ written backwards with the letter 1 swapped with the letter 2.

$$W_{n,m} = 13((13)^{n-1}(23)^{n-1})^m 2$$

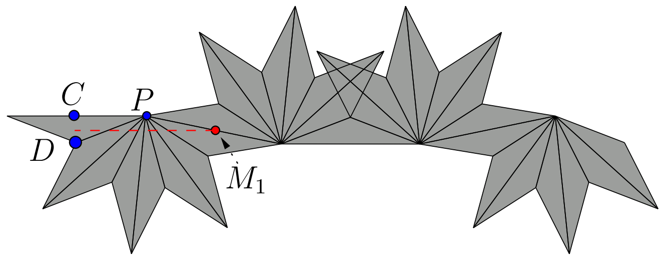

Figure 4: An unfolding for the word $Y_{4,1}$

But this is the same as $W_{n,m} = 1((31)^{n-1}(32)^{n-1})^m 32$. So $W_{n,m} = Y_{n,m}$.

Therefore, the unfolding of an isosceles triangle according to the word $Y_{n,m}$, exhibits this reflective symmetry. The axis of the glide reflection is preserved by this reflection but its orientation is reversed. The same is true for the long side of the first triangle of the unfolding. Therefore, this edge and the line preserved by the glide reflection must be parallel. $\diamondsuit$

A billiard path in an obtuse isosceles triangle that starts parallel to the long side and later hits the midpoint of the long side must close up by symmetry. These are exactly the type of paths we are looking for.

We will break the proof of theorem 2 into two cases. This is needed because the leading vertices are different in each case. The first case is easiest.

Lemma 7 *For each $n < x \leq n + \frac{1}{2}$ there is a periodic billiard path in T_x with combinatorics $Y_{n,1}$.*

Proof: Given the triangle T_x, unfold the triangle according to the word $Y_{n,1}$. There is a first midpoint of a long side, M_1, so that a rotation by π fixing M_1 preserves the unfolding. See figure 4. Let D be the obtuse vertex of the first triangle of the unfolding. Let C be the midpoint of the opposite side. By symmetry, it suffices to show that the line segment connecting M_1 to the segment $\overline{CD}$ orthogonally is contained in the unfolding.

The measure of angle CPM_1 is $2n\alpha = \frac{n\pi}{x} < \pi$ for $n < x$. Thus the orthogonal projection of M_1 to $\overline{CD}$ lies below C.

To see the orthogonal projection of M_1 lies above D, first note that the length $|PM_1|$ is strictly less than $|PD|$. Thus the orthogonal projection of

M_1 to $\overline{CD}$ lies above D so long as $2\angle CPD + \angle DPM_1 \geq \pi$. We evaluate the left hand side:

$$2\angle CPD + \angle DPM_1 = (2n+1)\alpha = \frac{(2n+1)\pi}{2x}$$

Therefore, the orthogonal projection of M_1 lies above D when $x \leq \frac{2n+1}{2}$. $\diamondsuit$

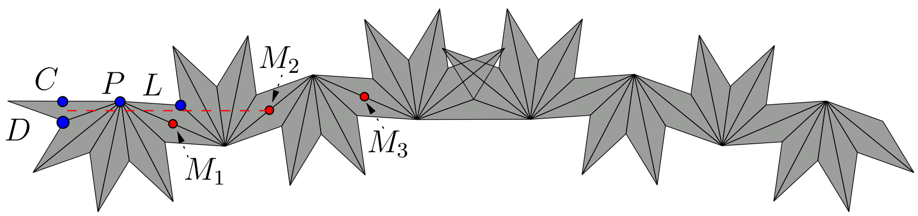

Figure 5: An unfolding for the word $Y_{4,2}$

The second case is more complicated. In fact our result is more easily proved non-constructively. A similar proof as the one given above could be given, but I believe this would make the argument far less clear. Consider the case of $Y_{n,2}$. A sample unfolding is shown in figure 5. Here, we would show that for some interval of isosceles triangles, the orthogonal projection of M_2 to the segment $\overline{CD}$ is entirely contained in the unfolding. Note that when the measure of angle $\angle CPL < \pi$, the point L becomes a new leader since it is below P. This occurs exactly when $x > n + \frac{1}{2}$ (or $\alpha < \frac{\pi}{2n+1}$). Thus, to show that $Y_{n,2}$ is a symbolic period for billiard path in a particular triangle, we would have to show that the orthogonal projection of M_2 to $\overline{CD}$ passes below L and above D.

Lemma 8 *For each $n + \frac{1}{2} < x < n+1$ there is a periodic billiard path in T_x with combinatorics $Y_{n,m}$ for some $m \in \mathbb{N}$.*

Proof: Consider the unfolding of T_x according to the infinite word $1(X_n)^\infty$. See figure 6. We will show that there is a M_m so that the orthogonal projection of M_m to $\overline{CD}$ passes below L and above D. A line through M_m and orthogonal to $\overline{CD}$ extends to a periodic billiard path with symbolic dynamics $Y_{n,m}$.

The crucial observation is that, because X_4 is the combinatorics of a periodic billiard path, there is an infinite rectangular strip contained in the unfolding $1(X_n)^\infty$. A beam of light shot orthogonally to $\overline{CD}$ passes below L and above D and then enters this strip. Once inside the strip, the beam of light will continue unobstructed until it hits the line containing $\{M_i\}$. We will show that the beam of light is wider than the vertical change between each M_i and M_{i+1}. It follows that some M_i must be hit by the beam of light. This path generates our desired periodic billiard path.

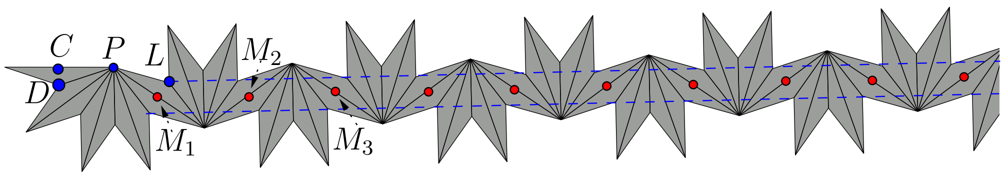

Figure 6: An unfolding for the word $1(X_4)^\infty$

Fix coordinates so that $P = (0,0)$. Then $D = (-\cos\alpha, -\sin\alpha)$ and $L = \Big(-\cos((2n+1)\alpha), -\sin((2n+1)\alpha)\Big)$. Let π_y be projection to the y-coordinate. Let f be the function which maps the angle α to the difference between the y coordinate of L and D.

$$f(\alpha) = \pi_y(L) - \pi_y(D) = \sin\alpha - \sin\Big((2n+1)\alpha\Big) \tag{4}$$

After some extensive trigonometry, we can reduce it into a more convenient form.

$$\begin{aligned} f(\alpha) &= \sin\alpha - (\sin\alpha\cos 2n\alpha + \sin 2n\alpha\cos\alpha) \\ &= \sin\alpha - \sin\alpha(\cos^2 n\alpha - \sin^2 n\alpha) - (2\sin n\alpha\cos n\alpha\cos\alpha) \\ &= 2\sin\alpha\sin^2 n\alpha - 2\sin n\alpha\cos n\alpha\cos\alpha \\ &= -2\sin n\alpha(-\sin\alpha\sin n\alpha + \cos n\alpha\cos\alpha) \\ &= -2\sin n\alpha\cos(n+1)\alpha \end{aligned} \tag{5}$$

If $\pi_y(D) < \pi_y(M_1) < \pi_y(L)$ then we can draw a segment connecting M_1 to the segment $\overline{CD}$ orthogonally. As in the previous lemma, this segment would extend to a periodic billiard path with symbolic period $Y_{n,1}$. Let us now check that $\pi_y(M_1) < \pi_y(L)$. We compute

$$\pi_y(M_1) = \frac{-\sin((2n+1)\alpha) - \sin((2n-1)\alpha)}{2}$$

So $\pi_y(M_1) < \pi_y(L)$ whenever

$$\sin\left((2n-1)\frac{\pi}{2x}\right) > \sin\left((2n+1)\frac{\pi}{2x}\right)$$

Which is certainly true when $n + \frac{1}{2} < x < n+1$ (these angles are close to π). We will therefore assume that $\pi_y(M_1) < \pi_y(D)$ because otherwise M_1 projects orthogonally to the interval $\overline{CD}$ and we would have a periodic billiard path with symbolic period $Y_{n,1}$.

Now we will use the fact that the word X_n is a periodic billiard path in T_x. In the unfolding of the word $1(X_m)^\infty$, the vector $M_{i+1} - M_i$ is invariant under choice of i. Further $M_{i+1} - M_i$ travels in direction $(n+1)\alpha - \frac{\pi}{2}$. We denote the distance from M_i to M_{i+1} by l. Thus, $\pi_y(M_i)$ is an evenly-spaced increasing sequence. Let $g(\alpha) = \pi_y(M_{i+1}) - \pi_y(M_i)$ which depends on α. Using equation 3 we see

$$\begin{aligned} g(\alpha) &= l\sin((n+1)\alpha - \tfrac{\pi}{2}) \\ &= \Big(\sin((n-2)\alpha) + \sin(n\alpha)\Big)\Big(-\cos((n+1)\alpha)\Big) \\ &= -\cos((n+1)\alpha)(2\cos\alpha\sin((n-1)\alpha)) \\ &= -2\cos\alpha\sin((n-1)\alpha)\cos((n+1)\alpha) \end{aligned} \tag{6}$$

If $g(\alpha) < f(\alpha)$ then there must be an M_m which projects to $\overline{CD}$ under L as desired. From the formulas above we see that in fact,

$$\frac{f(\alpha)}{g(\alpha)} = \frac{\sin(n\alpha)}{\cos\alpha\sin((n-1)\alpha)} \tag{7}$$

The graph of this function of α is depicted in figure 7. When $n+\frac{1}{2} < x < n+1$, then $\frac{\pi}{2n+2} < \alpha < \frac{\pi}{2n+1}$. Therefore

$$0 < (n-1)\alpha < n\alpha < \tfrac{\pi}{2} \quad \text{and} \quad \sin(n\alpha) > \sin((n-1)\alpha) \tag{8}$$

So that $\frac{f(\alpha)}{g(\alpha)} = \frac{\sin(n\alpha)}{\cos\alpha\sin((n-1)\alpha)} > \frac{1}{\cos\alpha} > 1$. $\diamondsuit$

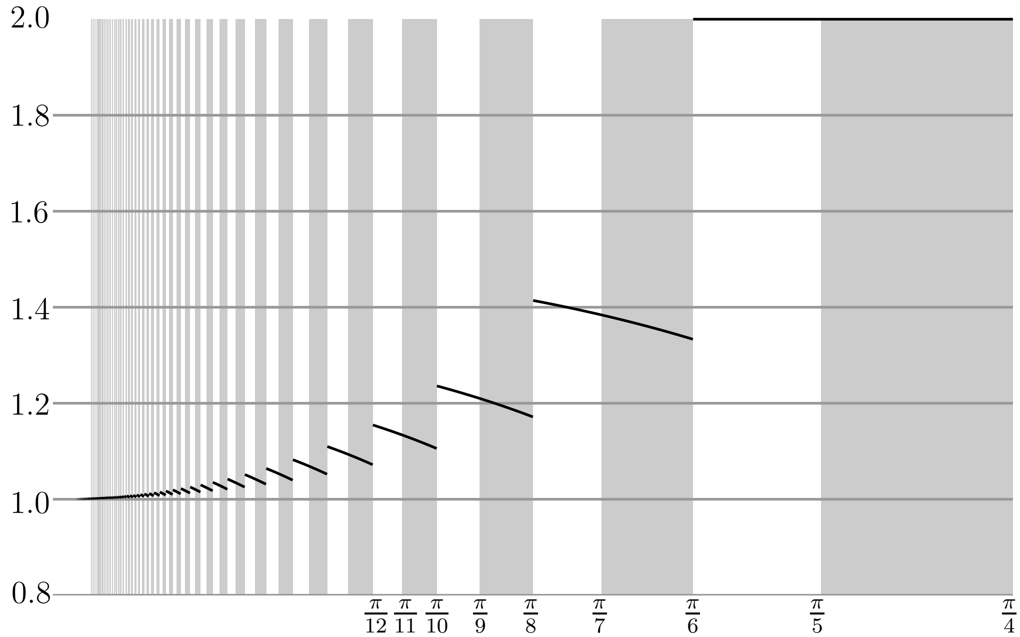

Figure 7: The graph of the ratio $\frac{f(\alpha)}{g(\alpha)}$ of equation 7 for $0 < \alpha < \frac{\pi}{4}$. We take the integer $n = \mathit{floor}(\frac{\pi}{2\alpha})$. The white regions are the values of α we consider in lemma 8.