AN INFINITE SURFACE WITH THE LATTICE PROPERTY I: VEECH GROUPS AND CODING GEODESICS

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ABSTRACT. We study the symmetries and geodesics of an infinite translation surface which arises as a limit of translation surfaces built from regular polygons, studied by Veech. We find the affine symmetry group of this infinite translation surface, and we show that this surface admits a deformation into other surfaces with topologically equivalent affine symmetries. The geodesics on these new surfaces are combinatorially the same as the geodesics on the original.

In this paper, we begin a systematic study of the geometric and dynamical properties of the surface S_1 shown below. This surface arises from a limit of surfaces built from two affinely regular n-gons as $n \to \infty$.

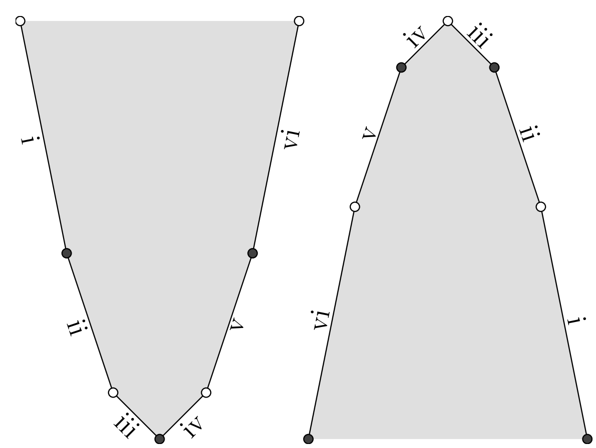

FIGURE 1. The surface S_1 is built from two infinite polygons in the plane: The convex hulls of the sets $\{(n, n^2) \ : \ n \in \mathbb{Z}\}$ and $\{(n, -n^2) \ : \ n \in \mathbb{Z}\}$. Roman numerals indicate edges glued by translations.

This study is motivated by work of Veech which shows that surfaces built in a similar manner from two regular polygons have special geometric and dynamical properties. See the original work of Veech [Vee89] or the survey [MT02]. In short, these surfaces exhibit affine symmetries analogous to the action of $SL(2, \mathbb{Z})$ on the square torus.

The surface S_1 also has affine symmetries described by a lattice in $SL(2, \mathbb{R})$. Furthermore, we will explain how S_1 arises from a limit of Veech's surfaces built from regular polygons. However, previous geometric and dynamical theorems on such surfaces do not directly apply to S_1 because this surface has infinite area, infinite genus, and two cone singularities with infinite cone angle. This motivates the question: Does the infinite genus surface S_1 exhibit

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"nice" geometric and dynamical properties? The purpose of this sequence of articles is to explain that S_1 has many such nice properties.

In this paper, we show the following:

- We show that the orientation preserving affine symmetry group is the congruence two subgroup of $\Gamma_2 \subset SL(2, \mathbb{Z})$.
- We describe the group of affine automorphisms of S_1. That is, the group of homeomorphisms $S_1 \to S_1$ which preserve the affine structure of S_1.
- We find a deformation of the surface S_1 given by $c \mapsto S_c$ for $c \geq 1$. This deformation has the property that each affine automorphism of S_1 is isotopic to an affine automorphism of S_c.
- We show that S_1 and S_c have isotopic geodesics for all $c > 1$.

This paper is structured as follows. In the next section, we construct the surfaces S_c, and explain how they relate to regular polygons. In section 2, we provide background on the subject of translation surfaces and Veech's work. In section 3, we give rigorous statements of the results mentioned above. We spend the remainder of the paper proving these statements.

1. The limiting process

Here is a dynamical way to describe a regular n-gon. Consider the rotation given by

$$R_t = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix} \in SO(2, \mathbb{R}).$$

The regular n-gon is the convex hull of the orbit of the point $(1,0)$ under the group generated by the rotation $R_{\frac{2\pi}{n}}$.

In order to take a limit we conjugate this rotation by the affine transform $S_t : (x, y) \mapsto (\frac{y}{\sin t}, \frac{x-1}{\cos t - 1})$. The purpose of S_t is to normalize three vertices of the polygons. We have

$$S_t(1,0) = (0,0), \; S_t(\cos t, \sin t) = (1,1), \text{ and } S_t(\cos t, -\sin t) = (-1,1).$$

Setting $c = \cos t$ and defining $T_c = S_t \circ R_t \circ S_t^{-1}$ yields the affine map $T_c : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$T_c : (x,y) \mapsto \big(cx + (c-1)y + 1, (c+1)x + cy + 1\big). \tag{1}$$

Let Q_c^+ be the convex hull of the set of points $\{P_c^k = T_c^k(0,0)\}_{k \in \mathbb{Z}}$. For $c = \cos \frac{2\pi}{n}$, Q_c^+ is an affinely regular n-gon. For $c = 1$ the collection of forward and backward orbits of $(0,0)$ is the set of points $\{(n, n^2) \mid n \in \mathbb{Z}\}$, the integer points on the parabola $y = x^2$. Finally for $c > 1$,

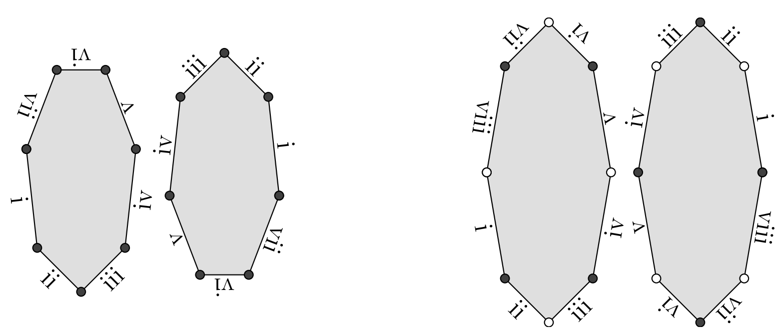

Figure 2. The translation surface $S_{\cos\frac{2\pi}{7}}$ and $S_{\cos\frac{\pi}{4}}$ are built from pairs of affinely regular polygons.

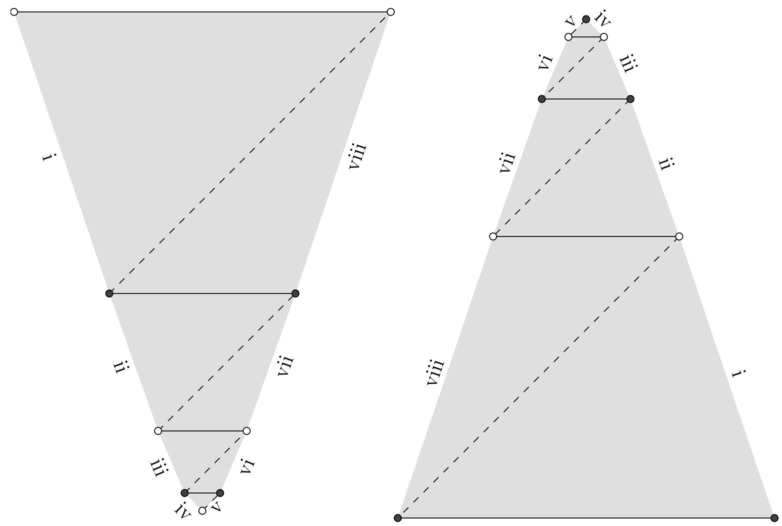

FIGURE 3. The surface S_c with $c = \frac{5}{4}$ is shown with some geodesic segments joining singularities.

the orbit of $(0, 0)$ lies on a hyperbola. Assume $c = \cosh t$. Up to an affine transformation, the orbit of $(0, 0)$ is $\{(\cosh nt, \sinh nt) \mid n \in \mathbb{Z}\}$.

We will use Q_c^+ to build our translation surfaces. Let Q_c^- be the image of Q_c^+ under a rotation by π around the origin. Each edge in Q_c^+ is parallel to its image in Q_c^-. We identify each edge of Q_c^+ to its image edge in Q_c^- by translation (rather than rotation). We call the resulting translation surface S_c. See figure 2 for some of the cases with $c < 1$. The case S_1 is drawn in figure 1, and $S_{\frac{5}{4}}$ is shown in figure 3.

Observe that for each k, the map $c \mapsto P_c^k = T_c^k(0, 0)$ is continuous. For this reason, we can think of the surface S_1 as a limit of the surfaces $S_{\cos \frac{2\pi}{n}}$ as $n \to \infty$ and $\cos \frac{2\pi}{n} \to 1$. Similarly, we view $c \mapsto S_c$ for $c \geq 1$ as a continuous deformation of translation surfaces. Concretely, we have the following:

Proposition 1 (A family of homeomorphisms)**.** *There is a family of homeomorphisms $h_{c,c'} : S_c \to S_{c'}$ defined for $c \geq 1$ and $c' \geq 1$ which satisfy the following statements.*

- *$h_{c,c}$ is the identity map, and $h_{c,c'} \circ h_{c',c''} = h_{c,c''}$.*
- *$h_{c,c'}$ sends singular points to singular points.*
- *$h_{c,c'}(Q_c^+) = Q_{c'}^+$ and $h_{c,c'}(Q_c^-) = Q_{c'}^-$.*
- *Let B be the bundle of the surfaces with singularities removed, $S_c \smallsetminus \Sigma$, over the ray $\{c \ : \ c \geq 1\}$. This bundle is taken to be locally isometric to $\mathbb{R}^3$. The map $B \times \{c' \ : \ c' \geq 1\} \to B$ which sends the pair consisting of a point $x \in S_c$ and a $c' \geq 1$ to the point $h_{c,c'}(x) \in S_{c'}$ is continuous in the metric topology.*

Proof. To construct such a family of maps, we triangulate each $Q_c^\pm$ in the same combinatorial way, and then define $h_{c,c'}$ piecewise, so that it affinely maps triangles to triangles. □

2. Background

Here we will briefly introduce some essential ideas in the subject of translation surfaces. See [MT02] for more detail.

A *translation surface* S is a collection of polygons in the plane with edges glued pairwise by translations. Any point in the interior of a polygon or in the interior of an edge has a neighborhood with an injective coordinate chart to the plane, which is canonical up to post composition with a translation. The vertices of the polygons can be cone singularities with cone angle which is an integer multiple of 2π. Infinite cone angles can arise if infinitely many polygons are used (as for S_1).

Suppose S is a translation surface and $\mathbf{u} \in \mathbb{R}^2$ is a unit vector. The *straight-line flow* on S in the direction $\mathbf{u}$ is the flow $F^t_{\mathbf{u}}$ given in local coordinates by

$$F^t_{\mathbf{u}}(x,y) = (x,y) + t\mathbf{u}.$$

Our translation surfaces will always be complete, so the straight-line flow of a point is defined for all time unless the orbit hits a singularity.

Let S and S' be translation surfaces. A homeomorphism $\widehat{A} : S \to S'$ is called an *affine* if in each local coordinates chart ψ is of the form

$$\psi(x,y) = (ax + by + t_1, cx + dy + t_2) \text{ with } A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in GL(2,\mathbb{R}) \text{ and } t_1, t_2 \in \mathbb{R}.$$

The constants t_1 and t_2 may depend on the chart. Because the transition functions are translations, the matrix A is an invariant of ψ. We call this matrix the *derivative*, $\mathbf{D}(\psi) = A \in GL(2,\mathbb{R})$.

There is a natural action of $GL(2,\mathbb{R})$ on translation surfaces. If $A \in SL(2,\mathbb{R})$ and S is a translation surface, we define $A(S)$ by composing each coordinate chart with the corresponding linear map $A : \mathbb{R}^2 \to \mathbb{R}^2$.

An *affine automorphism* of a translation surface S is an affine homeomorphism $\widehat{A} : S \to S$. The collection of all affine automorphisms of S form a group, called the affine automorphism group $\mathit{Aff}(S)$. The group $\mathbf{D}\big(\mathit{Aff}(S)\big) \subset GL(2,\mathbb{R})$ is called the *Veech group* of S and is denoted $\Gamma(S)$. An alternate definition of the Veech group is given by

$$\Gamma(S) = \{A \in GL(2,\mathbb{R}) \ : \ \exists \text{ an affine homeomorphism } \psi : A(S) \to S \text{ with } \mathbf{D}(\psi) = I\}.$$

When S has finite area, $\Gamma(S) \subset \widehat{SL}(2,\mathbb{R})$, the group of 2×2 matrices of determinant ± 1. In this case, we will say that S has the *lattice property* if $\Gamma(S)$ has finite covolume in $\widehat{SL}(2,\mathbb{R})$. In the compact case, this has the following strong consequence.

Theorem 2 (Veech Dichotomy [Vee89, §2])**.** *If S is a compact translation surface with the lattice property, then for all unit vectors $\mathbf{u}$ exactly one of the following holds.*

- *The straight line flow $F^t_{\mathbf{u}}$ is completely periodic (all non-singular trajectories are periodic).*
- *The straight line flow $F^t_{\mathbf{u}}$ is uniquely ergodic (there is only one invariant probability measure).*

3. Results

The following theorem describes the Veech groups of S_c.

Theorem 3 (Veech groups)**.** *The Veech groups $\Gamma(S_c) \subset GL(2,\mathbb{R})$ for $c \geq 1$ are generated by the involutions $-I$,*

$$A_c = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \quad B_c = \begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix}, \quad \text{and} \quad C_c = \begin{bmatrix} -c & c-1 \\ -c-1 & c \end{bmatrix}.$$

For $c = \cos(\frac{2\pi}{n})$, it is a theorem of Veech that the elements given above generate $\Gamma(S_c)$ [Vee89], which is an $(\frac{n}{2}, \infty, \infty)$-triangle group when n is even, and an $(n, 2, \infty)$ triangle group when n is odd. We describe the relations in this matrix group below.

Note in particular, the surface S_1 has the lattice property:

Corollary 4. *The orientation preserving part of* $\Gamma(S_1)$ *is the congruence two subgroup of* $SL(2, \mathbb{Z})$.

For all c, the matrices A_c, B_c, and C_c are involutions and act as reflections in geodesics in $\mathbb{H}^2$ when projectivized to elements of $Isom(\mathbb{H}^2) \cong PGL(2, \mathbb{R})$. By the theorem, the groups $\Gamma(S_c)$ are all representations of the group

$$\mathcal{G}^{\pm} = (\mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2) \oplus \mathbb{Z}_2 = \langle A, B, C, -I \mid A^2 = B^2 = C^2 = I \rangle.$$

The geodesics associated to A_c and C_c intersect at angle $\frac{2\pi}{n}$ when $c = \cos(\frac{2\pi}{n})$, are asymptotic when $c = 1$, and disjoint and non-asymptotic for $c > 1$. See figure 4.

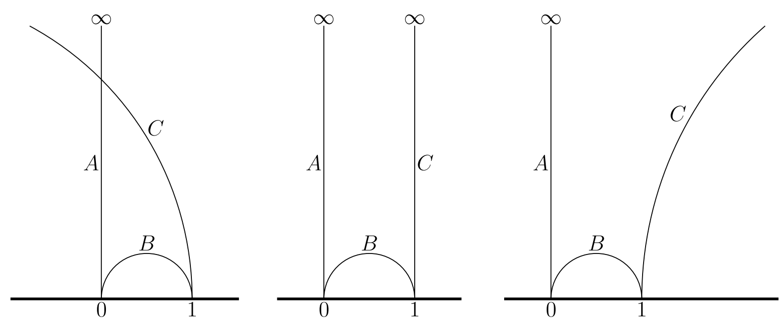

FIGURE 4. This figure shows the geodesics in the upper half plane model of $\mathbb{H}^2$ that A_c, B_c, and C_c reflect in for $c = \cos\frac{\pi}{4}$, $c = 1$, and $c = \frac{5}{4}$ from left to right.

When $c \geq 1$, it is clear that the triangle formed by the reflecting geodesics of A_c, B_c, and C_c is a fundamental domain for the action of $\Gamma(S_c)$, so the representations $\mathcal{G}^{\pm} \to \Gamma(S_c) \subset GL(2, \mathbb{R})$ are faithful when $c \geq 1$. When n is even and $c = \cos(\frac{2\pi}{n})$, the triangle formed by the reflecting geodesics is again a fundamental domain. Thus, $\Gamma(S_c)$ is isomorphic to $\mathcal{G}^{\pm}$ modulo the relation $(A_c C_c)^{\frac{n}{2}} = -I$. When n is odd, the element $(A_c C_c)^{\lfloor \frac{n}{2} \rfloor} A_c$ reflects in a geodesic orthogonal to the reflecting geodesic of B_c. In this case $\Gamma(S_c)$ is isomorphic to $\mathcal{G}^{\pm}$ modulo the relations $(A_c C_c)^n = I$ and $[(A_c C_c)^{\lfloor \frac{n}{2} \rfloor} A_c, B_c] = -I$.

Proposition 5. *For all* $c \geq 1$*, the map* $\mathbf{D} : \mathit{Aff}(S_c) \to \Gamma(S_c)$ *is a bijection.*

Because of this proposition, an affine automorphism is uniquely determined by its derivative. This allows us to introduce the following notation:

Notation 6. Recall $\Gamma(S_c) \cong \mathcal{G}^{\pm}$ when $c \geq 1$. Given $G \in \mathcal{G}^{\pm}$ and $c \geq 1$, we denote the corresponding element of $\Gamma(S_c) \subset GL(2, \mathbb{R})$ by G_c. Whenever the derivative map $\mathbf{D} : \mathit{Aff}(S) \to \Gamma(S)$ is a bijection, given $A \in \Gamma(S)$, we use $\widehat{A} \in \mathit{Aff}(S)$ to denote the corresponding affine automorphism $\widehat{A} : S \to S$.

We explicitly describe the topological action of generators for the affine automorphism group $\mathit{Aff}(S_c)$ in Lemma 10. The following theorem uses the family of homeomorphisms $h_{c,c'} : S_c \to S_{c'}$ in Proposition 1 to say that the affine automorphism groups act on each S_c

in the same way. Note that because the singular points of S_c are infinite cone singularities, any homeomorphism $S_c \to S_{c'}$ must map singularities to singularities. In particular, when two maps $S_c \to S_{c'}$ are isotopic, they are isotopic by an isotopy which preserves singularities.

Theorem 7 (Isotopic Affine Actions)**.** *The homeomorphisms $S_c \to S'_c$ given by $h_{c,c'} \circ \widehat{G}_c$ and $\widehat{G}_{c'} \circ h_{c,c'}$ are isotopic for all $G \in \mathcal{G}^{\pm} = \Gamma(S_c)$, $c \geq 1$ and $c' \geq 1$.*

The remaining results explain that the surfaces have the same geodesics in a combinatorial sense. A *saddle connection* in a translation surface S is a geodesic segment joining singularities with no singularities in its interior. We say two saddle connections σ and τ are *disjoint* if $\sigma \cap \tau$ is contained in the set of endpoints.

Theorem 8 (Isotopic triangulations)**.** *Suppose $\{\sigma_i\}_{i \in \Lambda}$ is a disjoint collection of saddle connections in S_c for $c \geq 1$ which triangulate the surface. Then for each $c' \geq 1$, there is a disjoint collection of saddle connections $\{\tau_i\}_{i \in \Lambda}$ and a homeomorphism $S_c \to S'_c$ isotopic to $h_{c,c'}$ so that $\sigma_i \mapsto \tau_i$ for all $i \in \Lambda$.*

We can use this theorem to show that all geodesics are the same combinatorially. To make this rigorous, say that an *interior geodesic* in a translation surface S is a continuous map $\gamma_0 : \mathbb{R} \to S$ so that $\gamma_0(\mathbb{R})$ contains no singularities, and in local coordinates $\gamma_0(a+t) = \gamma_0(a) + t\mathbf{u}$ where $\mathbf{u} \in \mathbb{R}^2$ is a unit vector. We call $\mathbf{u}$ the *direction* of γ_0. We say $\gamma : \mathbb{R} \to S$ is a *geodesic* if it is an interior geodesics or a pointwise limit of interior geodesics. Geodesics have directions; for a sequence of interior geodesics to converge pointwise, the directions must converge. The point of this construction is to make the space of geodesics closed.

Let $\{\sigma_i\}_{i \in \Lambda}$ be a collection of disjoint collection of saddle connections in S_c which triangulate S_c for $c \geq 1$. Suppose γ_0 is an interior geodesic in S_c for which $\gamma_0(0) \in \bigcup_{i \in \Lambda} \sigma_i$. Because S_c is complete and triangulated, the set $X = \gamma_0^{-1}\big(\bigcup_{i \in \Lambda} \sigma_i\big) \subset \mathbb{R}$ is discrete and bi-infinite. So, there is a unique increasing bijection $\psi : \mathbb{Z} \to X$ so that $\psi(0) = \gamma_0(0)$. The *coding of* γ_0 is the bi-infinite sequence $\langle e_n \in \Lambda \rangle_{n \in \mathbb{Z}}$ so that $\gamma_0 \circ \psi(n) \in \sigma_{e_n}$. Since γ_0 is interior, this sequence is unique. If γ is a limit of interior geodesics $\gamma_{0,k}$ satisfying $\gamma_{0,k}(0) \in e_0$ then the coding of γ is the limit of the codings of $\gamma_{0,k}$. (This limit makes sense because the singularities of S_c are not removable. For all $n \in \mathbb{Z}$ the sequence $k \mapsto \gamma_{0,k} \circ \psi(n)$ is eventually constant.) Every geodesic on S_c can be reparameterized so that it is an interior geodesic with $\gamma_0(0) \in \bigcup_{i \in \Lambda} \sigma_i$, or is a limit of such interior geodesics.

Let $\Omega_c \subset \Lambda^{\mathbb{Z}}$ denote the collection of codings of geodesics on S_c by the triangulation $\{\sigma_i\}_{i \in \Lambda}$. This set is shift invariant and closed, and thus a shift space. While it is a shift on a countable alphabet, each symbol can be followed by only two other symbols.

The above theorem indicates that for $c' \geq 1$ there is a corresponding triangulation $\{\tau_i\}_{i \in \Lambda}$ of $S_{c'}$. We can use this triangulation to code geodesics on $S_{c'}$ and construct a new shift space $\Omega_{c'} \subset \Lambda^{\mathbb{Z}}$.

Theorem 9 (Same geodesics)**.** *For each $c \geq 1$ and $c' \geq 1$, the two shift spaces Ω_c and $\Omega_{c'}$ are equal as subsets of $\Lambda^{\mathbb{Z}}$.*

In other words, if γ is a geodesics on S_c, then there is a geodesic on S'_c with the same coding, and vice versa.

The proofs of Theorems 8 and 9 hold more generally for deformations of translation surfaces with certain properties. See Lemmas 16 and 18. In particular, the same conclusions hold for a family of surfaces related to the infinite staircase. See [HHW10].

4. The affine automorphisms

In this section, we find and describe elements of the affine automorphism groups of the surfaces S_c defined in the previous section. At this point, we cannot assume Theorem 3, which described the generators of the Veech group. So, we use $\mathcal{G}_c^{\pm} \subset GL(2,\mathbb{R})$ to denote the group generated by $-I$, A_c, B_c, and C_c. Our description of affine automorphisms implies that that $\mathcal{G}_c^{\pm} \subset \Gamma(S_c)$. It also proves the Isotopic Affine Action Theorem, assuming $G_c \in \Gamma(S_c)$ implies $G_c \in \mathcal{G}_c^{\pm}$ (which turns out to be true).

To ensure our notation for affine automorphisms used in the previous section makes sense we must prove Proposition 5.

Proof of Proposition 5. Suppose $\psi \in \mathit{Aff}(S_c)$ satisfies $\mathbf{D}(\psi) = I$. Then, ψ maps saddle connections to saddle connections, and preserves their slope and length. In each surface S_c, there is only one saddle collection of slope one with length $\sqrt{2}$. Therefore, ψ must fix all points on this saddle connection. Since ψ fixes a non-singular point and $\mathbf{D}(\psi) = I$, ψ must be the identity map. □

It is useful to work with alternate generators for $\mathcal{G}^{\pm}$. Define the elements $D = BA$, $E = (-I)CB$. The elements $\{A, D, E, -I\}$ also are generators for $\mathcal{G}^{\pm}$. The corresponding matrices in $\mathcal{G}_c^{\pm}$ are given by

$$D_c = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \quad E_c = \begin{bmatrix} -c & c+1 \\ -c-1 & c+2 \end{bmatrix} \tag{2}$$

Note that D_c and E_c are orientation preserving parabolics.

Lemma 10 (Affine Automorphisms)**.** *For $c \geq 1$, $\mathcal{G}_c^{\pm} \subset \Gamma(S_c)$. Moreover, the affine automorphisms corresponding to generators of $-I, A_c, D_c, E_c \in \mathcal{G}_c^{\pm}$ may be described topologically (up to isotopy) as follows.*

- *$\widehat{-I}_c$ swaps the two pieces Q_c^+ and Q_c^- of S_c, rotating each piece by π.*
- *$\hat{A}_c$ is the automorphism induced by the Euclidean reflection in the vertical line $x = 0$, which preserves the pieces Q_c^+ and Q_c^- of S_c and preserves the gluing relations.*
- *$\hat{D}_c$ preserves the decomposition of S_c into maximal horizontal cylinders, and acts as a single right Dehn twist in each cylinder.*
- *$\hat{E}_c$ preserves the decomposition of S_c into maximal cylinders of slope 1, and acts as a single right Dehn twist in each cylinder.*

Remark 11. *The automorphism $\hat{F}_c$ corresponding to the element*

$$F_c = C_c A_c = \begin{bmatrix} c & c-1 \\ c+1 & c \end{bmatrix}$$

may be of special interest. The action of $\hat{F}_c$ preserves the decomposition into two pieces, Q_c^+ and Q_c^-. It acts on the top piece as T_c acts on the plane. (See equation 1). When $c \geq 1$, the surface S_c decomposes into a countable number of maximal strips in each eigendirection of F_c. The action of $\hat{F}_c$ preserves this decomposition into strips. We number each strip by integers, so that each strip numbered by n is adjacent to the strips with numbers $n \pm 1$. This numbering can be chosen so that the action of $\hat{F}_c$ sends each strip numbered by n to the strip numbered $n+1$. So, the action of $\hat{F}_c$ for $c \geq 1$ is as nonrecurrent as possible. Given any compact set $K \subset S_c \setminus \Sigma$, there is an N so that for $n > N$, $\hat{F}_c^n(K) \cap K = \emptyset$.

We begin by stating a well known result that gives a way to detect parabolic elements inside the Veech group. The idea is that a Dehn twist may be performed in a cylinder by a parabolic. See figure 5. A cylinder is a subset of a translation surface isometric to $\mathbb{R}/k\mathbb{Z} \times [0, h]$. The ratio $\frac{h}{k}$ is called the *modulus* of the cylinder.

Proposition 12 (Veech [Vee89, §9])**.** *Suppose a translation surface has a decomposition into cylinders $\{C_i\}_{i \in \Lambda}$ in a direction θ. Suppose further there is a real number $m \neq 0$ such that for every cylinder C_i, the modulus of M_i of C_i satisfies $mM_i \in \mathbb{Z}$. Then, there is an affine automorphism of the translation surface preserving the direction θ, fixing each point on the boundary of each cylinder, and acting as an mM_i power of single right Dehn twist in each cylinder C_i. The derivative this affine automorphism is the parabolic*

$$R_\theta \circ \begin{bmatrix} 1 & m \\ 0 & 1 \end{bmatrix} \circ R_\theta^{-1},$$

where $R_\theta \in SO(2)$ rotates the horizontal direction to direction θ.

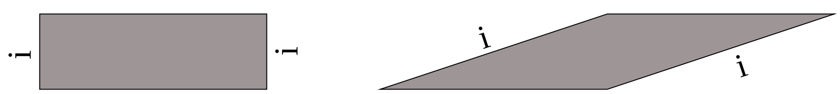

Figure 5. The right cylinder is obtained by applying a shear to the left cylinder. There is an affine homeomorphism from the right cylinder to the left with derivative I. The composition of these maps is used in Proposition 12.

In the cases of $\widehat{D}_c$ and $\widehat{E}_c$, each M_i will be equal, hence we get an affine automorphism which acts by a single right Dehn twist in each cylinder.

Proof of Lemma 10. Recall, the surface S_c for $c \geq 1$ was built from two pieces Q_c^+ and Q_c^-. We defined Q_c^+ to be the convex hull of the vertices $P_i = T_c^i(0,0)$ for $i \in \mathbb{Z}$, with T_c as in equation 1. Next Q_c^- was defined to be Q_c^+ rotated by π. S_c is built by gluing the edges of Q_c^+ to its image under Q_c^- by parallel translation. Indeed, it is obvious from this definition that the rotation by π which swaps Q_c^+ and Q_c^- restricts to an affine automorphism of the surface, $\widehat{-I}_c \in \mathit{Aff}(S_c)$. The derivative of $\widehat{-I}_c$ is $-I_c = -I$, which therefore lies in $\Gamma(S_c)$.

Now we will see that the reflection in the line $x = 0$ induces an affine automorphism $(\hat{A})$. The reflection is given the map $r : (x, y) \mapsto (-x, y)$. Q_c^+ is preserved because $r(P_i) = P_{-i}$, which follows from the fact that $r \circ T_c \circ r^{-1} = T_c^{-1}$. The reflection acts in the same way on Q_c^-, and thus preserves gluing relations. Thus, $\hat{A}_c$ is an affine automorphism and its derivative, A_c, lies in the Veech group.

We will show that each cylinder in the horizontal cylinder decomposition has the same modulus, which will prove that $\hat{D}_c$ lies in the affine automorphism group by proposition 12. Let $P_i = (x_i, y_i)$. The circumference of the $n - th$ cylinder numbered vertically is given by $C_n = 2x_{n-1} + 2x_n$, and the height is $H_n = y_n - y_{n-1}$. Now let $(x_{n-1}, y_{n-1}) = (\hat{x}, \hat{y})$, so that by definition of T_c, we have $(x_n, y_n) = (c\hat{x} + (c-1)\hat{y} + 1, (c+1)\hat{x} + c\hat{y} + 1)$. This makes

$$C_n = 2(c+1)\hat{x} + 2(c-1)\hat{y} + 2 \quad \text{and} \quad H_n = (c+1)\hat{x} + (c-1)\hat{y} + 1.$$

So that the modulus of each cylinder is $\frac{1}{2}$. It can be checked that the parabolic fixing the horizontal direction and acting as a single right Dehn twist in cylinders of modulus $\frac{1}{2}$ is given by D_c.

It is not immediately obvious that there is a decomposition into cylinders in the slope 1 direction. To see this, note that there is only one eigendirection corresponding to eigenvalue -1 of the $SL(2,\mathbb{R})$ part of the affine transformation

$$U : (x, y) \mapsto (-cx + (c-1)y + 1, -(c+1)x + cy + 1)$$

is the slope one direction. It also has the property that $U \circ T_c \circ U^{-1} = T_c^{-1}$, which can be used to show that U swaps P_i with P_{1-i}. Therefore segment $\overline{P_{1-i}P_i}$ always has slope one. The n-th slope one cylinder is formed by considering the union of trapezoid obtained by taking the convex hull of the points P_n, P_{n+1}, P_{1-n} and P_{-n} and the same trapezoid rotated by π inside Q_c^-. Now we will show that the moduli of these cylinders are all equal. The circumference and height of the n-th cylinder in this direction is given below.

$$C_n = \sqrt{2}(x_n - x_{1-n} + x_{n+1} - x_{-n})$$

$$H_n = \frac{\sqrt{2}}{2}(x_{n+1} - x_n, y_{n+1} - y_n) \cdot (-1, 1)$$

Let $P_n = (\hat{x}, \hat{y})$. Then $P_{n+1} = (c\hat{x} + (c-1)\hat{y} + 1, (c+1)\hat{x} + c\hat{y} + 1)$, $P_{1-n} = ((-c)\hat{x} + (c-1)\hat{y} + 1, (-c-1)\hat{x} + c\hat{y} + 1)$ and $P_{-n} = (-\hat{x}, \hat{y})$. We have

$$C_n = \sqrt{2}(2c+2)\hat{x} \quad \text{and} \quad H_n = \sqrt{2}\hat{x}.$$

The modulus of each cylinder is $\frac{1}{2c+2}$. Thus by proposition 12, $\hat{E}_c$ lies in the affine automorphism group. Again, we leave it to the reader to check that the derivative of $\hat{E}_c$ must be E_c. □

We now prove the Isotopic Affine Actions Theorem, assuming Theorem 3 which classifies the Veech group.

Proof of Theorem 7. It is enough to prove the statement for the generators $-I, A, D, E \in \mathcal{G}^{\pm}$. Let G be one of these generators. It can be observed that the affine actions $\widehat{G}_c : S_c \to S_c$ act continuously on the bundle B of surfaces S_c over $\{c \ : \ c \geq 1\}$. We must show that $h_{c,c'} \circ \widehat{G}_c$ and $\widehat{G}_{c'} \circ h_{c,c'}$ are isotopic. Let $c'' \geq 1$ be a number between c and c'. Consider the map $\phi_{c''} : S_c \to S'_c$ given by $\phi_{c''} = h_{c'',c'} \circ \widehat{G}_{c''} \circ h_{c,c''}$. Continuously moving c'' from c to c' yields the desired isotopy. □

5. A classification of saddle connections

In this section, we will classify the directions in S_c where saddle connections can appear. We begin with S_1.

We use the notation $\frac{p}{q} \equiv \frac{r}{s} \pmod 2$ to say that once the fractions are reduced to $\frac{p'}{q'}$ and $\frac{r'}{s'}$ so that numerator and denominator are relatively prime, we have $p' \equiv r' \pmod 2$ and $q' \equiv s' \pmod 2$. We use $\frac{p}{q} \not\equiv \frac{r}{s} \pmod 2$ to denote the negation of this statement.

In the statement of the following proposition, we use the concept of the holonomy of a saddle connection. Given any path $\gamma : [0,1] \to S$ in a translation surface which avoids the singularities on $(0,1)$, there is a *development* of γ into the plane. This is a curve $dev(\gamma) : [0,1] \to \mathbb{R}^2$ up to post-composition by with a translation, defined by following the local charts from S to the plane. The *holonomy vector* $hol(\gamma)$ is obtained by subtracting the endpoint of $dev(\gamma)$ from its starting point. The quantity $hol(\gamma)$ is invariant under homotopies which fix the endpoints. The notions of holonomy and the developing map are common in the world of (G, X) structures; see section 3.4 of [Thu97], for instance.

Proposition 13 (Saddle connections of S_1)**.** *Saddle connections $\sigma \subset S_1$ must have integral holonomy $hol_1(\sigma) \in \mathbb{Z}^2$. A direction contains saddle connections if and only if it has rational slope, $\frac{p}{q}$, with $\frac{p}{q} \not\equiv \frac{1}{0} \pmod 2$.*

Proof. The holonomy of a saddle connection must be integral, because the surface S_1 was built from two (infinite) polygons with integer vertices. The subgroup $\mathcal{G}_1^\pm \subset \Gamma(S_1)$ (generated by A_1, D_1, E_1, and $-I_1$) is the congruence two subgroup of $\widehat{SL}^\pm(2,\mathbb{Z})$. Thus, the linear action of $\mathcal{G}_1^\pm$ on the plane preserves the collection of vectors

$$RP = \{(p,q) \in \mathbb{Z}^2 \smallsetminus \{(0,0)\} \mid p \text{ and } q \text{ are relatively prime}\}.$$

Furthermore, the orbits of $(0,1)$, $(1,1)$, and $(1,0)$ under $\Gamma(S_1)$ are disjoint and cover RP. Thus, up to the affine automorphism group, the geodesic flow in a direction of rational slope looks like the geodesic flow in the horizontal, slope one, or vertical directions. There are saddle connections in both the horizontal and slope one directions, but not in the vertical direction. Therefore, rational directions contain saddle connections unless they are in the orbit of the vertical direction under $\mathcal{G}_1^\pm$. □

In order to make a similar statement for S_c, we will need to describe the directions that contain saddle connections. We will find it useful to note that there is a natural bijective correspondence between directions in the plane modulo rotation by π, and the boundary of the hyperbolic plane $\partial\mathbb{H}^2$. This can be seen group theoretically. Directions in the plane correspond to $\mathbb{S}^1 = SL(2,\mathbb{R})/\mathrm{H}$ where

$$\mathrm{H} = \{G \in SL(2,\mathbb{R}) \mid G\Big(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\Big) = \begin{bmatrix} \lambda \\ 0 \end{bmatrix} \text{ for some } \lambda > 0\}.$$

Both directions mod rotation by π and the boundary of the hyperbolic plane correspond to the real projective line, $\mathbb{RP}^1 = SL(2,\mathbb{R})/\mathrm{H}^\pm$, where

$$\mathrm{H}^\pm = \{G \in SL(2,\mathbb{R}) \mid G\Big(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\Big) = \begin{bmatrix} \lambda \\ 0 \end{bmatrix} \text{ for some } \lambda \neq 0\}.$$

Let $\mathbb{S}^1 = (\mathbb{R}^2 \smallsetminus \{(0,0)\})/\mathbb{R}_{>0}$, be the collection of rays leaving the origin. Consider the left action of the groups $\mathcal{G}_c^\pm$ on $\mathbb{S}^1$. We have the following.

Proposition 14 (Semi-conjugate actions)**.** *For all $c > 1$, there is a continuous (non-strictly) monotonic map $\varphi_c : \mathbb{S}^1 \to \mathbb{S}^1$ of degree one so that the following diagram commutes for all $G \in \mathcal{G}^\pm$.*

$$\begin{array}{ccc} \mathbb{S}^1 & \xrightarrow{G_c} & \mathbb{S}^1 \\ \downarrow \varphi_c & & \downarrow \varphi_c \\ \mathbb{S}^1 & \xrightarrow{G_1} & \mathbb{S}^1 \end{array}$$

We may also assume that ϕ_c preserves the horizontal ray $\{(x,0) : x > 0\}$ and the slope one ray $\{(x,x) : x > 0\}$. The map ϕ_c commutes with the rotation of the plane by π.

Proof. Existence of this map follows from [Ghy87], for instance. The following is a more natural proof using hyperbolic geometry.

Let $\mathcal{G}^+ \subset \mathcal{G}^\pm$ be those elements $G \in \mathcal{G}^\pm$ for which each $\det G_c = 1$. This is an index two subgroup and isomorphic to the product of the free group with two generators with $\mathbb{Z}/2\mathbb{Z}$. We let $\mathcal{G}_c^+ = \{G_c : G \in \mathcal{G}^+\} \subset SL(2,\mathbb{R})$. And use $P\mathcal{G}_c^+ \subset PSL(2,\mathbb{R})$ to denote the projectivized groups. The surfaces $\Sigma_c = \mathbb{H}^2/\mathcal{G}_c^+$ and $\Sigma_1 = \mathbb{H}^2/\mathcal{G}_1^+$ are thrice punctured spheres whose fundamental groups are canonically identified with $\mathcal{G}^+$. Let $\psi : \Sigma_c \to \Sigma_1$ be a homeomorphism which induces the trivial map between the fundamental groups (as identified with $\mathcal{G}^+$). We may choose ψ so that it is invariant under the action of $\mathcal{G}^\pm/\mathcal{G}^+$ (which acts on each surface as a reflective symmetry). Since the fundamental groups are identified, there is a canonical lift to a map between the universal covers $\widetilde{\psi} : \widetilde{\Sigma}_c \to \widetilde{\Sigma}_1$ so that $\psi \circ G = G \circ \psi$ for all $G \in \mathcal{G}^+$, where G is acting on the universal covers as an element of the covering group. This also holds for elements $G \in \mathcal{G}^\pm$ because of the $\mathcal{G}^\pm/\mathcal{G}^+$ invariance of ψ. Now noting the canonical identification of these universal covers with $\mathbb{H}^2$ we have $\widetilde{\psi} : \mathbb{H}^2 \to \mathbb{H}^2$ so that $\widetilde{\psi} \circ G_c = G_1 \circ \widetilde{\psi}$ for all $G \in \mathcal{G}^+$. This map induces a continuous monotonic degree 1 map on the boundary of the hyperbolic plane $\mathbb{RP}^1$. The desired map φ_c is a lift of this map to the double cover $\mathbb{S}^1$ of $\mathbb{RP}^1$. □

The map φ_c is reminiscent of the famous devil's staircase, a continuous surjective map $[0, 1] \to [0, 1]$ which contracts intervals in the compliment of a Cantor set to points. Indeed, the limit set Λ_c of the group $\mathcal{G}_c^\pm$ is a $\mathcal{G}_c^\pm$-invariant Cantor set, and the connected components of the domain of discontinuity, $\mathbb{RP}^1 \smallsetminus \Lambda_c$, are contracted to points by φ_c.

We will see that the saddle connections in S_c and in S_1 are topologically the same. We will now make this notion rigorous. Given a path $\gamma : [0, 1] \to S_c$, we use $[\gamma]$ to denote the equivalence class of paths which are homotopic to γ relative to their endpoints. We do not allow these homotopies to pass through singular points.

Theorem 15 (Classification of saddle connections)**.** *There is a saddle connection in direction $\theta \in \mathbb{S}^1$ on S_c for $c > 1$ if and only if there is a saddle connection in the direction $\varphi_c(\theta)$ on S_1. Equivalently, θ contains saddle connections if and only if θ is an image of the horizontal or slope one direction under an element of $\mathcal{G}_c^\pm = \langle -I_c, A_c, D_c, E_c\rangle$. Furthermore, the collection of homotopy classes containing saddle connections are identical in S_c and S_1. That is, for all saddle connections $\sigma \subset S_c$ there is a saddle connection in the homotopy class $[h_{c,1}(\sigma)]$ in S_1, and for all saddle connections $\sigma' \subset S_1$ there is a saddle connection in the homotopy class $[h_{1,c}(\sigma')]$ in S_c.*

We will prove this theorem by first proving a more abstract lemma. Then we will demonstrate that S_c and S_1 satisfy the conditions of the lemma. We need the following two definitions.

The *wedge product* between two vectors in $\mathbb{R}^2$ is given by

$$(a, b) \wedge (c, d) = ad - bc. \tag{3}$$

This is the signed area of the parallelogram formed by the two vectors.

The function *sign* $: \mathbb{R} \to \{-1, 0, 1\}$ assigns one to positive numbers, zero to zero, and -1 to negative numbers.

Lemma 16. *Let $h : S \to T$ be a homeomorphism between translation surfaces satisfying the following statements.*

1. *S admits a triangulation by saddle connections.*
2. *For every saddle connection $\sigma \subset S$ the homotopy class $[h(\sigma)]$ contains a saddle connection of T.*
3. *Every pair of saddle connections $\sigma_1, \sigma_2 \subset S$ satisfies*

$$sign\big(hol(\sigma_1) \wedge hol(\sigma_2)\big) = sign\big(hol(h(\sigma_1)) \wedge hol(h(\sigma_2))\big).$$

Then, for every saddle connection $\sigma \subset T$, the homotopy class $[h^{-1}(\sigma)]$ contains a saddle connection of S.

Proof. Let $\mathcal{T}_S$ be the triangulation of S by saddle connections given to us by item 1. By item 2, we we can straighten $h(\mathcal{T}_S)$ to a triangulation $\mathcal{T}_T$ of T by saddle connections.

We define the *complexity* of a saddle connection $\sigma \subset T$ relative to the triangulation $\mathcal{T}_T$ to be the number of times σ crosses a saddle connection in $\mathcal{T}_T$. We assign the saddle connections in $\mathcal{T}_T$ complexity zero. Supposing the conclusion of the lemma is false, there exists at least one saddle connection $\sigma \subset T$ so that $[h^{-1}(\sigma)]$ contains no saddle connection of S. We may choose such a saddle connection $\sigma \subset T$ so that it has minimal complexity with respect to $\mathcal{T}_T$. By the remarks above this minimal complexity must be at least one. The idea of the proof is to reduce the general case (arbitrary complexity) to a case similar to the complexity one case. In the complexity one case, σ passes through two triangles which form a convex quadrilateral.

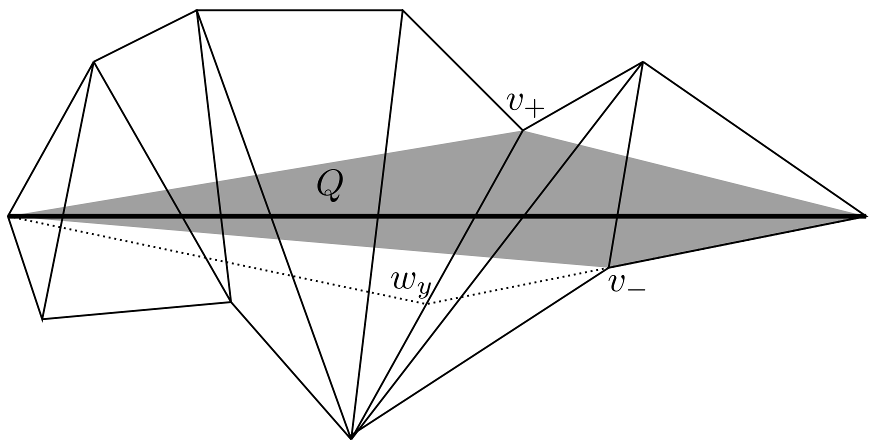

FIGURE 6. The saddle connection σ is developed into the plane along with the triangles in $\mathcal{T}_T$ that it intersects.

The saddle connection σ crosses through a sequence of triangles $\Delta_0, \ldots, \Delta_K$ of the triangulation $\mathcal{T}_T$. We may develop the saddle connection σ into the plane along with the triangles. We call the union of developed triangles the *unfolding* U. Without loss of generality, we may assume that the developed image of σ is horizontal and the y-coordinate of points in this image is zero. The sequence of saddle connections $\tau_1, \ldots, \tau_K \in \mathcal{T}_T$ crossed by σ develop to edges of the triangles with one endpoint above σ and one below σ. Call these endpoints top and bottom vertices, respectively.

We will find a convex quadrilateral Q built out of saddle connections in T and contained in the triangles crossed by σ. One diagonal of Q will be σ, and the edges of Q will be saddle connections with complexity less than that of σ. Two of the vertices of the developed image of Q must be the end points of σ. We choose one vertex v_+ of Q to be a top vertex with minimal y-coordinate. This choice guarantees that the convex hull (viewed in the development) of σ and v_+ is a triangle T_+ contained U. Let τ_i be a saddle connection of $\mathcal{T}_T$ crossed by σ with top endpoint v_+. For $y < 0$ let w_y be the point on τ_i with y-coordinate given by y. Consider the family of closed convex quadrilaterals Q_y obtained by taking the convex hull of v_+, σ and w_y. There is a largest $y < 0$ for which a bottom vertex v_- appears in Q_y. The convex hull of σ, v_+ and v_- is the desired quadrilateral Q.

The boundary of Q consists of four saddle connections $\nu_1, \ldots, \nu_4$ with complexity relative to $\mathcal{T}_T$ less than that of σ. Because we assumed σ had minimal complexity, there are saddle connections $\nu'_1, \ldots, \nu'_4 \subset S$ in the homotopy classes $[h^{-1}(\nu_1)], \ldots, [h^{-1}(\nu_4])]$ respectively. Finally, because of item 3, the saddle connections $\nu'_1, \ldots, \nu'_4$ in S must also form a strictly convex quadrilateral Q'. The quadrilateral Q' must then have diagonals, one of which lies in the homotopy class $[h^{-1}(\sigma)]$. See figure 7. □

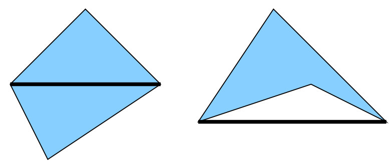

FIGURE 7. To destroy a diagonal of a quadrilateral, the quadrilateral must be made non-convex. This violates property 3 of lemma 16.

The following proposition implies the classification of saddle connections, Theorem 15.

Proposition 17. *The homeomorphism $h_{c,c'} : S_c \to S_{c'}$ satisfies the conditions of Lemma 16.*

Proof. It is sufficient to prove that $h_{1,c}$ satisfies the conditions of the lemma, because we can write $h_{c,c'} = h_{1,c}^{-1} \circ h_{1,c'}$. See Proposition 1. Note that if two homeomorphisms satisfy the lemma, then so does their composition. In addition, if a h satisfies the lemma, then so does h^{-1} (by the conclusion of the lemma applied to h). So, we will restrict to the case of $h_{1,c}$.

Item 1 is trivial. We leave it to the reader to triangulate S_1.

Item 2 follows from propositions 13. By proposition 13 all saddle connections of S_1 are the images of saddle connections in the horizontal and slope one directions under $\mathcal{G}_1^{\pm}$. Observe that that for each saddle connection τ in the horizontal and slope one directions that appears in S_1, there is a saddle connection the homotopy class $\tau' \in [h_{1,c}(\tau)]$. Let σ be an arbitrary saddle connection in S_1. Then, $\sigma = \widehat{G}_1(\tau)$ for some saddle connection τ of slope zero or one and some $G \in \mathcal{G}^{\pm}$ with $\widehat{G}_1$ denoting the corresponding affine automorphism. Let $\tau' \in [h_{1,c}(\tau)]$ be the corresponding saddle connection in S_c. Then by Theorem 7,

$$\sigma' = \widehat{G}_c(\tau') \in \widehat{G}_c([h_{1,c}(\tau)]) = [h_{1,c} \circ \widehat{G}_1(\tau)] = [h_{1,c}(\sigma)]$$

is the desired saddle connection in S_c.

Now we show item 3 holds. Let σ and σ' be saddle connections in the surface S_1. Let $\theta_0 = \{(x,0) \ : \ x > 0\}$ and $\theta_1 = \{(x,x) \ : \ x > 0\}$ be horizontal and slope one rays in $\mathbb{S}^1$. Then we can choose $\alpha, \alpha' \in \{\theta_0, \theta_1\}$ and $G_1, G'_1 \in \mathcal{G}_1^{\pm}$ such that the holonomies of these saddle connections satisfy $hol_1(\sigma) \in G_1(\alpha)$ and $hol_1(\sigma') \in G'_1(\alpha')$. It follows that the corresponding elements $G_c, G'_c \in \mathcal{G}_c^{\pm}$ satisfy $hol_c \circ h_{1,c}^{-1}(\sigma) \in G_c(\alpha)$ and $hol_c \circ h_{1,c}^{-1}(\sigma') \in G'_c(\alpha')$. We must prove that

$$sign\big(G_1(\alpha) \wedge G'_1(\alpha')\big) = sign\big(G_c(\alpha) \wedge G'_c(\alpha')\big),$$

where the sign of the wedge is computed using arbitrary representatives of the classes. This follows essentially from Proposition 14 which defined the the map $\varphi_c : \mathbb{S}^1 \to \mathbb{S}^1$. By this proposition, the statement above is equivalent to

$$sign\big(\phi_c \circ G_c(\alpha) \wedge \phi_c \circ G'_c(\alpha')\big) = sign\big(G_c(\alpha) \wedge G'_c(\alpha')\big).$$

Note that for any degree one continuous monotonically increasing map $\psi : \mathbb{S}^1 \to \mathbb{S}^1$ which commutes with rotation by π satisfies

$$sign\big(\psi(\beta) \wedge \psi(\beta')\big) \in \{0, sign(\beta \wedge \beta')\}$$

for every $\beta, \beta' \in \mathbb{S}^1$. In our setting, we have $G_c(\alpha) \wedge G'_c(\alpha') \neq 0$ if these two directions are fixed by different parabolic subgroups of $\mathcal{G}_c^{\pm}$. Note that if the directions $G_1(\alpha)$ and $G'_1(\alpha')$ are distinct, then they are fixed by different parabolic subgroups of $\mathcal{G}_1^{\pm}$. Then, by the commutative diagram in Proposition 14, the two directions $\phi_c \circ G_c(\alpha)$ and $\phi_c \circ G'_c(\alpha')$ are fixed by distinct parabolics subgroups of $\mathcal{G}_1^{\pm}$. Therefore $\phi_c \circ G_c(\alpha) \wedge \phi_c \circ G'_c(\alpha') \neq 0$. □

Now we prove Theorem 8, i.e. that the surfaces S_c and $S_{c'}$ admit the same triangulations.

Proof of Theorem 8. Let $\{\sigma_i\}_{i \in \Lambda}$ is a disjoint collection of saddle connections in S_c for $c \geq 1$ which triangulate the surface. By Proposition 17, the homeomorphism $h_{c,c'}$ satisfies the conditions of Lemma 16. So, we can find geodesics $\sigma'_i \in [h(\sigma_i)]$ for all i. The collection $\{\sigma'_i\}_{i \in \Lambda}$ is also a disjoint collection of saddle connections in $S_{c'}$ which triangulate the surface.

We define $h'_{c,c'} : S_c \to S'_c$ to be the homeomorphism which acts affinely on the triangles, and preserves the labeling of edges by Λ. We claim h' is isotopic to $h_{c,c'}$. Because Lemma 16 is satisfied for the map $h_{c,c'}$ for all pairs of surfaces, we can always do the above construction. Therefore, we can think of $h'_{c,c'}$ as well defined for all $c \geq 1$ and $c' \geq 1$, and this family of homeomorphisms satisfies the conclusions of Proposition 1. So, to see that $h_{c,c'}$ is isotopic to $h'_{c,c'}$, consider the isotopy given by $h_{c,c''} \circ h'_{c'',c'}$ as c'' varies between c and c'. □

Similar logic will apply to the proof of Theorem 9, which says that codes of geodesics in these surfaces are the same. This theorem follows from Proposition 17 together with the following lemma.

Lemma 18. *Suppose $h : S \to S'$ is a homeomorphism between translation surfaces satisfying the three statements of lemma 16. Let $\{\sigma_i\}_{i \in \Lambda}$ be a disjoint collection of saddle connections which triangulate of S and $\{\sigma'_i\}_{i \in \Lambda}$ be saddle connections of S' so that $\sigma'_i \in [h(\sigma_i)]$ for all $i \in \Lambda$. Then the shift spaces $\Omega, \Omega' \subset \Lambda^{\mathbb{Z}}$ which code geodesics using these triangulations in S and S', respectively, are equal.*

Proof. It suffices to prove that the same finite words appear in each space. We must show that if γ is a finite geodesic segment which hits no singularities, then there is another geodesic segment γ' in S' which intersects the relevant saddle connections in the same order.

Let γ be a finite trajectory in S. To derive a contradiction, it would have to be that γ crosses at least two saddle connections in $\{\sigma_i\}_{i \in \Lambda}$. Develop γ into the plane, along with the saddle connections it crosses. The sequence of saddle connections in the orbit-type develop to a sequence of segments s_i in the plane. We define $\mathcal{L}$ to be the space of all lines in $\mathbb{R}^2$ that cross through the interiors of each segment s_i. The *slalom hull* of the sequence $\langle s_i \rangle$ is

$$SH = \bigcup_{\ell \in \mathcal{L}} \ell.$$

Clearly this slalom hull is non-empty, because the line containing the developed image of γ lies in $\mathcal{L}$. The boundary of the SH consists of four rays and a finite set of segments that pull back to connections. Orient SH so that it is nearly horizontal as in the figure 8. Note that $\mathbb{R}^2 \smallsetminus SH$ consists of two convex components. We call the finite segments in the top component of ∂SH the *top chain*. Similarly, the segments in the bottom component will be called the bottom chain. Further, note there are *diagonals*, which are parallel to the infinite rays in ∂SH. They are formed by connecting the left-most vertex of the top chain to the right-most vertex of the bottom chain, and vice versa. We call the union of the top chain, the bottom chain, and the diagonals the *saddle chain*, and note that they form a loop.

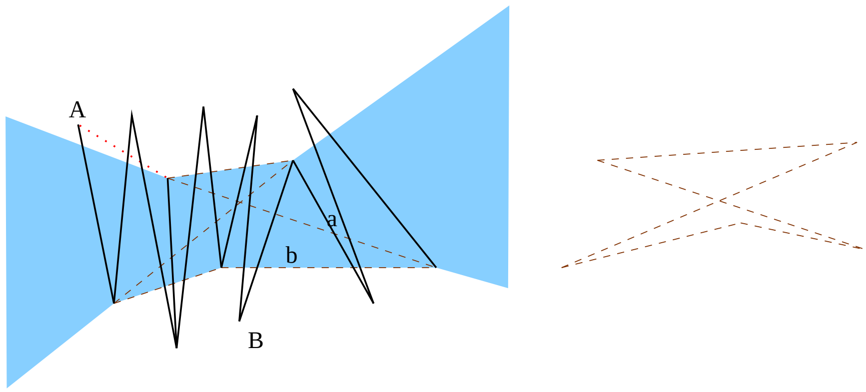

FIGURE 8. On the left, The slalom hull of a sequence of solid segments is rendered as the shaded polygon. The saddle chain is the sequence of dotted lines. The right shows a deformation of the slalom chain. No deformation of the slalom chain which preserves the cyclic ordering of edges in $\mathbb{RP}^2$ can destroy the slalom hull.

Each c_i in the saddle chain of S pulls back to a saddle connection σ_i of S. Let σ_i' be the corresponding saddle connections in S'. By item 3 of lemma 16, the directions of saddle connections σ_i' must be in the same cyclic order as the directions for σ_i. We may develop the saddle connections σ_i' into the plane to obtain the chain of segments c_i'. This chain of segments has essentially the same combinatorics. Consider the sequence of segments given by the diagonal running from top left to bottom right, and then moving across the top chain from left to right, and ending with the other diagonal. Each segment must be rotated slightly counterclockwise to reach the subsequent one. Thus, the region bounded by the top chain, by the ray leaving the top left vertex in the direction of the diagonal, and the ray leaving the top right vertex in the direction of the other diagonal must bound a convex set. Similarly, there is another natural convex set bounded by the lower chain and some rays. The convex sets may not intersect, because they are guaranteed to lie in two opposite quadrants of the division of the plane by lines through the diagonals. Thus, we have an analogous set of lines $\mathcal{L}'$ which pass through the diagonals and not the lower and upper chains. The slalom hull SH' given by the same formula is non-empty.

We claim that SH' is the slalom hull for the segments $\{s_i'\}$. This is equivalent to saying that no endpoint of a segment s_i' lies within SH', which will imply our theorem. Some of the locations of endpoints are determined, because they are endpoints of segments in the chain $\{c_i'\}$.

Suppose $B \in \mathbb{R}^2 \smallsetminus SH$ is a endpoint of a segment s_i in the development of S which crosses one of the edges in the slalom hull, $b \in \partial SH$. See figure 8. Let B' be the corresponding endpoint of the segment s_i' in the development from S'. Then B' cannot be in SH', lest it destroy the segment c_i', which must pull back to a saddle connection in S'. (Intersections between saddle connections are essential, in the sense that if two saddle connections intersect, then so must any pair of curves in the associated homotopy classes).

The more difficult case is when $A \in \mathbb{R}^2 \smallsetminus SH$ is an endpoint of a segment s_i in the development S which crosses one of the rays in ∂SH. Without loss of generality, assume the ray crossed is the upper left ray in the development. Let a be the diagonal element of the slalom chain that is parallel to this ray, and a' the corresponding element in the development of S'. We consider the canonical homotopy class of S, with the cone singularities removed, of paths joining the pull back of the top left vertex in ∂SH to the pull back A. This homotopy class should contain the path which develops to follow the ray of ∂SH leaving the top left vertex until it hits the segment s_i and then follows s_i to A. Let d be the infimum of the lengths of paths in this homotopy class. There is a limiting path $p \subset S$ that may pass through singularities, that achieves this infimum. This path consists of a sequence of saddle connections, and turns only rightward in total angle less than π. The initial segment of the developed image of p must immediately leave SH, hence its orientation when compared to a is determined. Let p' be the corresponding chain of saddle connections in S', which must likewise turn rightward by total angle less than π. The initial segment of the developed image of p' must immediately leave SH', because its orientation when compared to a' must match that of the case in S. Moreover, since p' only bends rightward, the path cannot return to SH'. This concludes the argument that SH' is the slalom hull for the segments $\{s_i'\}$. We know this contains lines, any of which contain a segment which pulls back to a trajectory γ' in S' with the same orbit type as γ in S. □

6. No other affine automorphisms

The last step to the proof of Theorems 3 and 7 is to demonstrate that all affine automorphisms of the surface lie in the group generated by the elements we listed.

Lemma 19. *All affine automorphisms of the surface S_c are contained in the group generated by $\widehat{-I}_c$, $\hat{A}_c$, $\hat{D}_c$, and $\hat{E}_c$.*

Proof. Let us suppose that for some $c \geq 1$ there is an $M \in GL(2, \mathbb{R})$ in the Veech group $\Gamma(S_c)$ and a corresponding element $\hat{M}$ in the affine automorphism group $\mathit{Aff}(S_c)$. We will prove that $\hat{M}$ lies in the group generated by the four elements $\widehat{-I}_c$, $\hat{A}_c$, $\hat{D}_c$, and $\hat{E}_c$.

Let $\theta = \{(x, 0) \ : \ x > 0\} \in \mathbb{S}^1$ be the horizontal direction. We know that the image $M(\theta)$ must contain the holonomies of saddle connections of S_c. Further more the horizontal and slope one directions can be distinguished, since the smallest area maximal cylinder in the horizontal direction has two cone singularities in its boundary, while the smallest area maximal cylinder in the slope one direction has four cone singularities in its boundary. Thus, by theorem 15, there must be an element $N_c \in \mathcal{G}_c^{\pm}$ satisfying $M(\theta) = N_c(\theta)$. It follows that $N_c^{-1} \circ M$ preserves the horizontal direction.

There must be a corresponding element element $\hat{N}_c^{-1} \circ \hat{M} \in \mathit{Aff}(S_c)$ with derivative $N_c^{-1} \circ M$. The automorphism must fix the decomposition into horizontal cylinders, and fix each cylinder in the decomposition (because the cylinders have distinct areas). The smallest area horizontal cylinder is isometric in each S_c. It is built from two triangles, the convex hull of $(0, 0)$, $(1, 1)$, and $(-1, 1)$ and the same triangle rotated by π, with diagonal sides of the first glued to the diagonal sides of the second by translation. $\hat{N}_c^{-1} \circ \hat{M} \in \mathit{Aff}(S_c)$ must preserve this cylinder and permute the pair of cone singularities in the boundary. Therefore

$$N_c^{-1} \circ M = \begin{bmatrix} 1 & 2n \\ 0 & 1 \end{bmatrix} = D_c^n \quad \text{or} \quad N_c^{-1} \circ M = \begin{bmatrix} 1 & -2n \\ 0 & -1 \end{bmatrix} = -I \circ A_c \circ D_c^n$$

for some $n \in \mathbb{Z}$. Therefore, $M = N_c \circ D_c^n$ or $M = N_c \circ -I \circ A_c \circ D_c^n$, all of which lie in $\mathcal{G}_c^{\pm}$. Therefore, by Proposition 5, the corresponding affine automorphism satisfies $\hat{M} = \hat{N}_c \circ \hat{D}_c^n$ or $\hat{M} = \hat{N}_c \circ \widehat{-I} \circ \hat{A}_c \circ \hat{D}_c^n$. $\square$

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