

DYNAMICS ON AN INFINITE SURFACE WITH THE LATTICE PROPERTY

W. PATRICK HOOPER

ABSTRACT. Dynamical systems on an infinite translation surface with the lattice property are studied. The geodesic flow on this surface is found to be recurrent in all but countably many rational directions. Hyperbolic elements of the affine automorphism group are found to be nonrecurrent, and a precise formula regarding their action on cylinders is proven. A deformation of the surface in the space of translation surfaces is found, which “behaves nicely” with the geodesic flow and action of the affine automorphism group.

In this paper, we study dynamical systems on an infinite translation surface with the lattice property. We build the surface S_1 by gluing together two polygonal parabolas as in figure 1. This surface is infinite in many respects; it has infinite genus, infinite area, and two cone points with infinite cone angle.

This study is motivated by the study of closed, finite area translation surfaces with the lattice property, a property which has been found to have great significance for dynamics. For instance, Veech has shown the geodesic flow on a finite area translation surface with the lattice property satisfies a dichotomy. In every direction, the geodesic flow is either completely periodic (decomposes into periodic cylinders), or minimal and uniquely ergodic. See [Vee89] or [MT02].

The surface S_1 may be obtained from some of Veech’s original examples by a limiting process. See section 2. Moreover, theorem 1 demonstrates that S_1 has the lattice property. The geodesic flow on the surface supports a trichotomy. In directions of rational slope, the flow is either completely periodic or highly nonrecurrent, decomposing into strips. In irrational directions, the flow is recurrent, but the flow contains no closed trajectories and no saddle connections. See section 4.

We study the dynamics of the action of the affine automorphism group in section 5. Our theorem governs the dynamics of hyperbolic elements, \hat{H} , of the affine automorphism group of S_1 . The action of \hat{H} on S_1 is not recurrent. Nonetheless, cylinders are distributed evenly, in the following sense. Theorem 30 states there is a positive constant κ_H

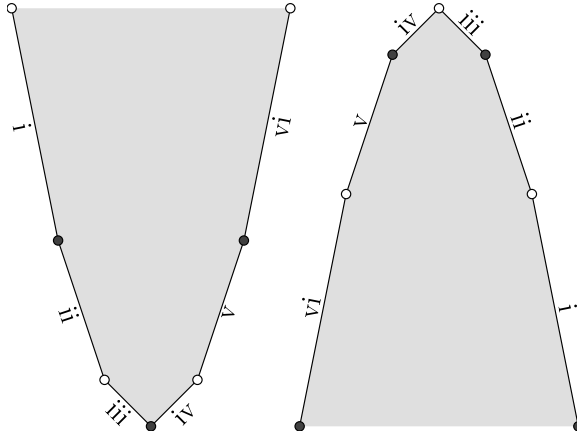


FIGURE 1. The translation surface S_1 is built from two infinite polygons, the convex hull of the points (n, n^2) for $n \in \mathbb{Z}$ and the convex hull of the points $(n, -n^2)$. Roman numerals indicate edges identified by translations.

depending only on \hat{H} so that given any two cylinders \mathcal{A} and \mathcal{B} on S_1 ,

$$(1) \quad \lim_{m \rightarrow \infty} m^{\frac{3}{2}} \text{Area}(\hat{H}^m(\mathcal{A}) \cap \mathcal{B}) = \kappa_H \text{Area}(\mathcal{A}) \text{Area}(\mathcal{B}).$$

The theorem relies on the understanding of the action of the affine automorphism group on homology built up in section 5. The constant κ_H has a precise definition, related to billiards in hyperbolic triangles, and can be explicitly computed for any given \hat{H} . See theorem 27 and appendix 6.2.

The pair consisting of our surface and its affine automorphism group, $(S_1, \text{Aut}(S_1))$, exhibits a canonical deformation along a path $(S_c, \text{Aut}(S_c))$ parameterized by real numbers $c \geq 1$. Figure 4 shows a surface S_c with $c > 1$. The translation surfaces S_c are all canonically homeomorphic to S_1 . The groups $\text{Aut}(S_c)$ are canonically isomorphic to $\text{Aut}(S_1)$. The action of $\text{Aut}(S_c)$ on S_c is the same as the action of $\text{Aut}(S_1)$ on S_1 , up to conjugacy by the canonical homeomorphism and isotopy. See section 3.

The deformation of S_1 preserves a natural coding of geodesic trajectories as well. We discover that the surfaces S_c for $c \geq 1$ each have the same saddle connections. That is, up to homotopy, the canonical homeomorphisms between the surfaces send saddle connections to saddle connections. See theorem 7. One way to code trajectories of the geodesic flow is to triangulate the surface by cutting along disjoint saddle connections. A trajectory is coded by the sequence of these saddle connections it crosses. Let $c_1, c_2 \geq 1$. Since S_{c_1} and S_{c_2} have

the same saddle connections, it makes sense to compare the coding of trajectories. In section 3.2, we show that every coding of a trajectory in S_{c_1} also appears as a coding for a trajectory in S_{c_2} .

A brief background on translation surfaces is provided in the next section. In section 2, we construct our infinite surfaces. The introduction to section 3 states theorems describing the affine automorphism and Veech groups of our surfaces.

The remainder of the paper may be divided into three paths, each dependent only on the descriptions of the affine automorphism and Veech groups. The reader is invited to take the path of her choice. Continuing along section 3 will result in proofs that affine automorphism and Veech groups are as described in the theorems, as well as proofs that the surfaces have the same saddle connections and codings of trajectories. A walk through section 4 will prove that the geodesic flow in irrational directions on the surface S_1 is recurrent. A stroll through section 5 will uncover a detailed understanding of the action of the affine automorphism groups on homology, which is used to prove the asymptotic formula given in equation 1 above. Proofs of facts in section 5 that require meandering have been moved to the appendix.

The author would like to thank Matt Bainbridge, Francois Ledrappier, Rich Schwartz, Yaroslav Vorobets, Barak Weiss, and Amie Wilkinson for helpful discussions involving this work.

1. TRANSLATION SURFACES

Translation surfaces can be defined in many different ways. We choose the point of view of polygons in \mathbb{R}^2 up to translation. If P and Q are polygons in \mathbb{R}^2 , we say P and Q are equivalent, $P \sim Q$, if there is a translation taking the vertices of P to the vertices of Q and preserving the counterclockwise cyclic ordering. We will always use the counterclockwise orientation on the edges of our polygons.

A *translation surface* is a surface built by identifying edges of polygons. Our edge identifications must be made by translations. Two edges may be identified only if they are parallel, have the same length, and opposite orientations. A simple example is the square torus, the surface obtained from a square by identifying opposite edges by translations. If our surface is built from finitely many polygons, we insist that it be closed, and when built from countably many polygons, we insist that it be complete (every Cauchy sequence must converge). In general, the surface is not locally isometric to \mathbb{R}^2 . At places where the vertices of polygons are identified there may be cone singularities. When built from finitely many polygons, a translation surface's cone angles must

be integer multiples of 2π . In the infinite case, there may be infinite cone angles. We use Σ to denote the set of all cone singularities.

There are always many ways to build the same translation surface from polygons. We say two translation surfaces S and S' are *the same* if they may be cut into equivalent polygons which are identified by translations in the same combinatorial way. This is equivalent to the existence of a direction preserving isometry from S to S' . Here we use the metric and notion of direction induced by regarding our polygons as subsets of \mathbb{R}^2 . Because translation surfaces are built by gluing together pieces of \mathbb{R}^2 by translations, which preserve distance and direction, both notions are natural on translation surfaces. The notion of direction may be viewed as a geodesic flow invariant fibration of the unit tangent bundle of the surface S over the circle, which restricts to a trivial bundle on $S \setminus \Sigma$.

There is a natural action of the affine group on translation surfaces. We consider the group $GL(2, \mathbb{R})$ acting on \mathbb{R}^2 in the usual way, by matrix multiplication on column vectors. If S is obtained by gluing polygons P_1, P_2, \dots together by translations determined by edge identifications, then for $A \in GL(2, \mathbb{R})$ the surface $A(S)$ is obtained by gluing $A(P_1), A(P_2), \dots$ by the same edge identifications. This operation constructs a new translation surface, because affine transformations send parallel lines to parallel lines, preserve ratios of lengths of parallel segments, and uniformly preserve or reverse orientations. In a possible abuse of notation, there is a natural map $S \rightarrow A(S)$ induced by the maps $A : P_i \rightarrow A(P_i)$, which we also denote by $A : S \rightarrow A(S)$.

The *Veech group* of a translation surface S is the subgroup of $\Gamma(S) \subset GL(2, \mathbb{R})$ so that if $A \in \Gamma(S)$ then $A(S)$ and S are the same, in the sense above. This tells us that there exists at least one direction preserving isometry from $\varphi_A : A(S) \rightarrow S$. The collection of all maps $\hat{A} = \varphi_A \circ A : S \rightarrow S$ with $A \in \Gamma(S)$ is a group known as the *affine automorphism group* of S , $Aut(S)$. The map $\mathbf{D} : Aut(S) \rightarrow \Gamma(S)$ given by $\mathbf{D} : \hat{A} \mapsto A$ is called the derivative map because \hat{A} acts on the tangent plane of a non-singular point of S as the action of A on \mathbb{R}^2 .

When S is finite area, elements of the Veech group $\Gamma(S)$, must be area preserving and hence $\Gamma(S) \subset SL^\pm(2, \mathbb{R})$. Here we use $SL^\pm(2, \mathbb{R})$ to denote the space of all 2×2 matrices with determinant ± 1 . So long that S is built from a finite number of polygons, Veech showed that $\Gamma(S)$ is a discrete subgroup of $SL^\pm(2, \mathbb{R})$. A surface is said to have the *lattice property* if $\Gamma(S)$ is a lattice in $SL^\pm(2, \mathbb{R})$.

The surfaces we consider in this paper have infinite area. Thus, it is possible that there are elements of the Veech group with determinants

other than ± 1 . It is also possible that the Veech group is indiscrete. In the infinite cases we consider, we will prove that neither of these occur.

2. A FAMILY OF TRANSLATION SURFACES

Veech considered translation surfaces built from regular n -gons [Vee89]. Indeed, every surface built from finitely many regular n -gons has the lattice property.

Here is a dynamical way to write a n -gon. Take the rotation of the plane given by the matrix

$$R_t = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix}.$$

The regular n -gon is the convex hull of the orbit of the point $(1, 0)$ under the group generated by $R_{\frac{2\pi}{n}}$.

In order to take a limit we conjugate this rotation by the affine transform $S_t : (x, y) \mapsto (\frac{y}{\sin t}, \frac{x-1}{\cos t-1})$. The purpose of S_t is to normalize the first three vertices of the polygons. We have

$$S_t(1, 0) = (0, 0), \quad S_t(\cos t, \sin t) = (1, 1), \quad \text{and} \quad S_t(\cos t, -\sin t) = (-1, 1).$$

Setting $c = \cos t$ and defining $T_{\cos t} = S_t \circ R_t \circ S_t^{-1}$ yields

$$(2) \quad T_c : (x, y) \mapsto (cx + (c-1)y + 1, (c+1)x + cy + 1)$$

Let Q_c^+ be the convex hull of the set of points $\{P_c^k = T_c^k(0, 0)\}_{k \in \mathbb{Z}}$. For $c = \cos \frac{2\pi}{n}$, Q_c^+ is an affinely regular n -gon. For $c = 1$ the collection of forward and backward orbits of $(0, 0)$ is the set of points $\{(n, n^2) \mid n \in \mathbb{Z}\}$, the integer points on the parabola $y = x^2$. Finally for $c > 1$, the orbit of $(0, 0)$ lies on a hyperbola. Assume $c = \cosh t$. Up to an element of the affine group, the orbit of $(0, 0)$ is $\{(\cosh nt, \sinh nt, 1) \mid n \in \mathbb{Z}\}$.

We will use Q_c^+ to build our translation surfaces. Let Q_c^- be the image of Q_c^+ under a rotation by π . Now identify each edge of Q_c^+ to its image edge in Q_c^- by *parallel translation*. We call the resulting translation surface S_c . See figure 2 for some of the cases with $c < 1$. The case S_1 is drawn in figure 1.

In the cases $c \geq 1$, the surfaces S_c are all homeomorphic. These surfaces have infinite genus and a pair of singular points with infinite cone angles. The homeomorphism from S_c to S_1 is induced by a homeomorphism $Q_c^+ \rightarrow Q_1^+$ which fixes the origin $(0, 0)$ and sends each edge of Q_c^+ to the corresponding edge of Q_1^+ , preserving ratios of distances on each edges. Conjugating by a rotation by π induces a similar map $Q_c^- \rightarrow Q_1^-$. Moreover, because these maps were assumed to preserve ratios of distances on each edge, the map extends to a homeomorphism $H_c : S_c \rightarrow S_1$.

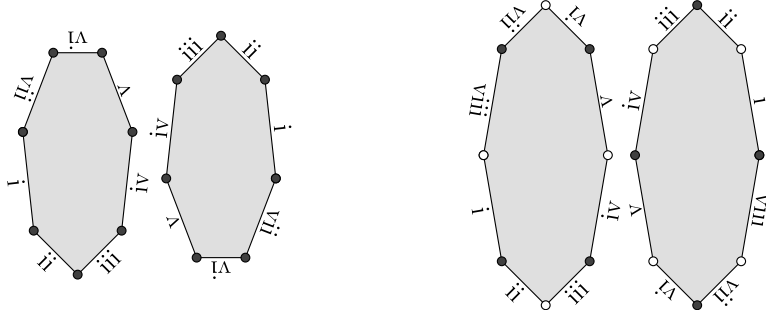


FIGURE 2. The translation surface $S_{\cos \frac{2\pi}{7}}$ and $S_{\cos \frac{\pi}{4}}$ are built from pairs of affinely regular polygons.

3. AFFINE AUTOMORPHISM GROUPS

In this section, we compute the affine automorphism groups of the surfaces S_c defined in the previous section. We cover the case $c = \cos \frac{2\pi}{n}$ without proof, since the groups are well known.

First, we will describe the Veech groups $\Gamma(S_c)$ for $c = \cos \frac{2\pi}{n}$ with $n \in \mathbb{Z}$. These groups always contain $-I_c = -I$ together with the following involutions.

$$A_c = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \quad B_c = \begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix}, \quad \text{and} \quad C_c = \begin{bmatrix} -c & c-1 \\ -c-1 & c \end{bmatrix}.$$

These elements generate a reflection group in a triangle with two ideal vertices and one vertex with an angle of $\frac{2\pi}{n}$. In the case of n even, these elements generate the Veech group. When n is odd, we must include an additional involution, which folds the triangle in half.

$$\begin{bmatrix} -\cos \frac{\pi}{n} & -\frac{\sin^2 \frac{\pi}{n}}{\cos \frac{\pi}{n}} \\ -\cos \frac{\pi}{n} & \cos \frac{\pi}{n} \end{bmatrix}$$

When n is odd, these elements generate a reflection group in a hyperbolic triangle with one ideal vertex, one right angle, and one angle of $\frac{\pi}{n}$. These groups are all lattices, thus these surfaces have the lattice property.

In our surfaces S_c with $c \geq 1$, the last automorphism makes no appearance. The Veech groups $\Gamma(S_c)$ for $c = \cos \frac{2\pi}{n}$ or $c \geq 1$ may be thought of as representations of the group

$$\mathcal{G}^\pm = (\mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2) \oplus \mathbb{Z}_2 = \langle A, B, C, -I \mid A^2 = B^2 = C^2 = I \rangle,$$

where I denotes the identity matrix. Given an element $G \in \mathcal{G}^\pm$, we will use G_c to denote the corresponding element of $\Gamma(S_c)$.

Theorem 1 (Veech groups). *The Veech groups $\Gamma(S_c) \subset GL(2, \mathbb{R})$ for $c \geq 1$ are generated by the involutions $-I_c = -I$,*

$$A_c = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \quad B_c = \begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix}, \quad \text{and} \quad C_c = \begin{bmatrix} -c & c-1 \\ -c-1 & c \end{bmatrix}.$$

We will use \mathcal{G}_c^\pm to denote the subgroup $\langle -I_c, A_c, B_c, C_c \rangle \subset SL(2, \mathbb{R})$. The theorem claims $\mathcal{G}_c^\pm = \Gamma(S_c)$ for $c \geq 1$.

These groups \mathcal{G}_c^\pm are best understood by looking at their action on the hyperbolic plane $\mathbb{H}^2 = SL(2, \mathbb{R})/SO(2, \mathbb{R})$. The elements A_c , B_c , and C_c act on \mathbb{H}^2 as reflections in geodesics. When $c = 1$, the triangle formed has three ideal vertices, and the Veech group $\Gamma(S_1)$ is precisely the congruence two subgroup of $SL^\pm(2, \mathbb{Z})$. This group is a lattice. Finally, when $c > 1$, the triangle formed has two ideal vertices and one ultra-ideal vertex. See figure 3.

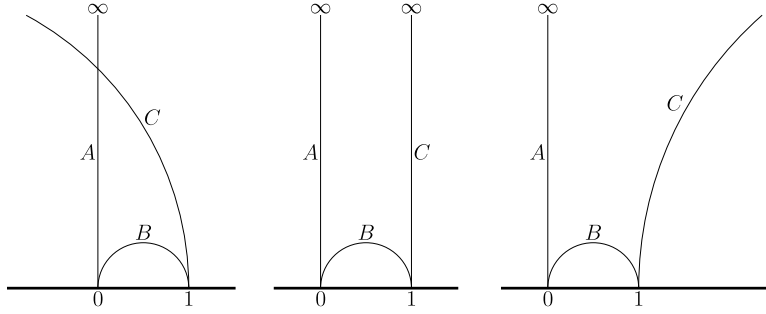


FIGURE 3. This figure shows the geodesics in the upper half plane model of \mathbb{H}^2 that A_c , B_c , and C_c reflect in for $c = \cos \frac{\pi}{4}$, $c = 1$, and $c = \frac{5}{4}$ from left to right.

We also wish to give a description of the affine automorphism groups $Aut(S_c)$. Before we do this, it is useful to work with alternate generators for \mathcal{G}^\pm . Define the elements $D = BA$, $E = (-I)CB$. The elements $\{A, D, E, -I\}$ also are generators for \mathcal{G}^\pm . From the above, we have

$$(3) \quad D_c = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \quad E_c = \begin{bmatrix} -c & c+1 \\ -c-1 & c+2 \end{bmatrix}$$

The affine automorphism group of a surface S may be generated by choosing for each generator G of $\Gamma(S)$ an element $\hat{G} \in Aut(S)$ so that $\mathbf{D}(\hat{G}) = G$, and by including generators for the kernel of $\mathbf{D} : Aut(S) \rightarrow \Gamma(S)$.

We will now describe the affine automorphism groups of S_c for $c = \cos \frac{2\pi}{n}$. The surfaces S_c are affinely equivalent to gluings of a pair of regular polygons n -gons. The surfaces built from a pair of regular

n -gons have several affine automorphisms whose derivatives are Euclidean rotations and reflections. The affine automorphism groups include conjugates of these Euclidean elements. The groups also contain the automorphism \hat{D}_c , which applies a single right Dehn twist to each horizontal cylinder, and \hat{E}_c , which acts by a single right Dehn twist in every slope one cylinder. When n is odd, these elements generate the affine automorphism group and $\ker(\mathbf{D})$ is trivial. For n even, we also need to include the affine automorphism which swaps the two polygons by translation. This involution generates $\ker(\mathbf{D})$ when n is even.

The surfaces S_c for $c \geq 1$ also admit cylinder decompositions into horizontal and vertical cylinders. See figure 4.

Theorem 2 (Affine Automorphisms). *The subgroup $\ker(\mathbf{D}) \subset \text{Aut}(S_c)$ is trivial for $c \geq 1$. Generators for $\text{Aut}(S_c)$ with $c \geq 1$ may be described as follows.*

- $\widehat{-I}_c$ swaps the two pieces of S_c , rotating each piece by π .
- \hat{A}_c is the automorphism induced by the Euclidean reflection in the vertical line $x = 0$, which preserves the pieces Q_c^+ and Q_c^- of S_c and preserves the gluing relations.
- \hat{D}_c preserves the decomposition of S_c into maximal horizontal cylinders, and acts as a single right Dehn twist in each cylinder.
- \hat{E}_c preserves the decomposition of S_c into maximal cylinders of slope 1, and acts as a single right Dehn twist in each cylinder.

Let $\hat{\mathcal{G}}_c$ denote the group of self-homeomorphisms up to isotopy generated by $\langle \widehat{-I}_c, \hat{A}_c, \hat{D}_c, \hat{E}_c \rangle$. The theorem states that these generators may be realized by affine automorphisms and generate $\text{Aut}(S_c)$. Both these theorems will be proved in the subsequent subsections.

Recall, the translation surfaces S_c for $c \geq 1$ are canonically homeomorphic. Further, by the above theorem, the action of the affine automorphism groups $\text{Aut}(S_c)$ on S_c are topologically the same. That is, they are the same up to conjugacy by this homeomorphism and isotopies fixing the singularities. We use this point of view in the proofs.

Remark 3. *The automorphism \hat{F}_c corresponding to the element*

$$F_c = C_c A_c = \begin{bmatrix} c & c-1 \\ c+1 & c \end{bmatrix}$$

may be of special interest. The action of \hat{F}_c preserves the decomposition into two pieces, Q_c^+ and Q_c^- . It acts on the top piece as the element of T_c acts on the plane. (See equation 2). When $c \geq 1$, the surface S_c decomposes into a countable number of maximal strips in each eigendirection of F_c . The action of \hat{F}_c preserves this decomposition into strips.

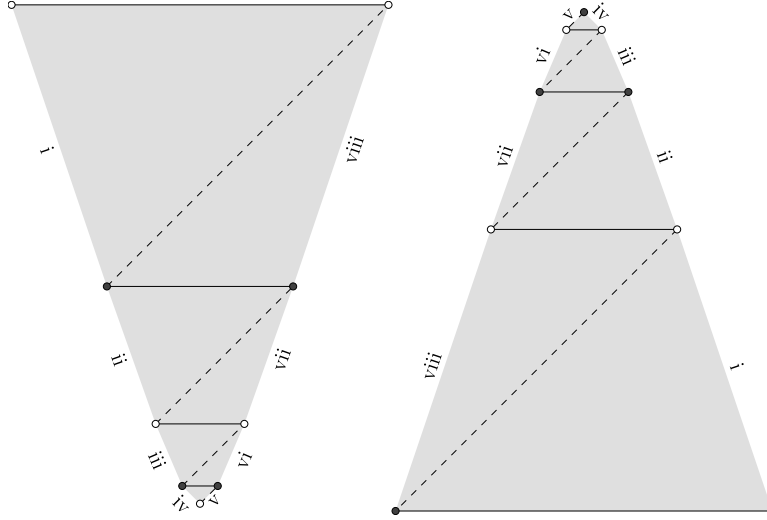


FIGURE 4. The surface S_c with $c = \frac{5}{4}$ is shown. Horizontal and slope one saddle connections are drawn.

We number each strip by integers, so that each strip numbered by n is adjacent to the strips with numbers $n \pm 1$. There are two possible numberings satisfying this condition. If the correct numbering system is chosen, the action of \hat{F}_c sends each strip numbered by n to the strip numbered $n + 1$.

The action of \hat{F}_c for $c \geq 1$ is as nonrecurrent as possible. Given any compact set $K \subset S_c \setminus \Sigma$, there is an N so that for $n > N$, $\hat{F}_c^n(K) \cap K = \emptyset$.

The affine automorphism theorem implies the Veech group theorem, up to computations of the derivatives of the affine automorphisms. In the next subsection, we show that our list of self-homeomorphisms \widehat{I}_c , \widehat{A}_c , \widehat{D}_c , and \widehat{E}_c are indeed realized by affine automorphisms. It is more difficult to show that every affine automorphism lies in the generated group. This will be demonstrated in the subsequent subsections.

3.1. Generators of the affine automorphism groups. In this section we prove that the affine automorphism and Veech groups contain the elements we listed in theorems 1 and 2 in the cases of $c \geq 1$.

We begin by stating a well known result that gives a way to detect parabolic elements inside the Veech group. The idea is that a Dehn twist may be performed in a cylinder by a parabolic. See figure 5. The height of a Euclidean cylinder divided by its circumference is called the *modulus* of the cylinder.

Proposition 4 (Veech). *Suppose a translation surface has a decomposition into cylinders $\{C_i\}_{i \in \Lambda}$ in a direction θ . Suppose further there is a number m such that for every cylinder C_i , the modulus of M_i of C_i satisfies $m/M_i \in \mathbb{Z}$. Then there is an affine automorphism of the translation surface preserving the direction θ , fixing each point on the boundary of each cylinder, and acting as an m/M_i power of single right Dehn twist in each cylinder C_i . The derivative this affine automorphism is a parabolic fixing direction θ .*



FIGURE 5. A right Dehn twist in a cylinder may be realized by a parabolic (or shear). The action on the cylinder preserves boundary points, but twists the interior of the cylinder.

In our cases, each M_i will be equal, hence we get an affine automorphism which acts by a single right Dehn twist in each cylinder.

Lemma 5. *Let $c \geq 1$. The subgroup $\mathcal{G}_c^\pm = \langle -I_c, A_c, D_c, E_c \rangle \subset SL(2, \mathbb{R})$ is contained in the Veech group $\Gamma(S_c)$. The self-homeomorphisms of S_c given by $\widehat{-I}_c$, \widehat{A}_c , \widehat{D}_c , and \widehat{E}_c may be realized by affine automorphisms, and thus generate a subgroup of $Aut(S_c)$.*

Proof. Recall, the surface S_c for $c \geq 1$ was built from two pieces Q_c^+ and Q_c^- . We defined Q_c^+ to be the convex hull of the vertices $P_i = T_c^i(0, 0)$ for $i \in \mathbb{Z}$, with T_c as in equation 2. Next Q_c^- was defined to be Q_c^+ rotated by π . S_c is built by gluing the edges of Q_c^+ to its image under Q_c^- by parallel translation. Indeed, it is obvious from this definition that the rotation by π which swaps Q_c^+ and Q_c^- restricts to an affine automorphism of the surface, $\widehat{-I}_c \in Aut(S_c)$. The derivative of $\widehat{-I}_c$ is $-I_c = -I$, which therefore lies in $\Gamma(S_c)$.

Now we will see that the reflection in the line $x = 0$ induces an affine automorphism (\widehat{A}). The reflection is given the map $r : (x, y) \mapsto (-x, y)$. Q_c^+ is preserved because $r(P_i) = P_{-i}$, which follows from the fact that $r \circ T_c \circ r^{-1} = T_c^{-1}$. The reflection acts in the same way on Q_c^- , and thus preserves gluing relations. Thus, \widehat{A}_c is an affine automorphism and its derivative, A_c , lies in the Veech group.

We will show that each cylinder in the horizontal cylinder decomposition has the same modulus, which will prove that \widehat{D}_c lies in the affine automorphism group by proposition 4. Let $P_i = (x_i, y_i)$. The

circumference of the n -th cylinder numbered vertically is given by $C_n = 2x_{n-1} + 2x_n$, and the height is $H_n = y_n - y_{n-1}$. Now let $(x_{n-1}, y_{n-1}) = (\hat{x}, \hat{y})$, so that by definition of T_c , we have $(x_n, y_n) = (c\hat{x} + (c-1)\hat{y} + 1, (c+1)\hat{x} + c\hat{y} + 1)$. This makes

$$C_n = 2(c+1)\hat{x} + 2(c-1)\hat{y} + 2 \quad \text{and} \quad H_n = (c+1)\hat{x} + (c-1)\hat{y} + 1.$$

So that the modulus of each cylinder is $\frac{1}{2}$. It can be checked that the parabolic fixing the horizontal direction and acting as a single right Dehn twist in cylinders of modulus $\frac{1}{2}$ is given by D_c .

It is not immediately obvious that there is a decomposition into cylinders in the slope 1 direction. To see this, note that there is only one eigendirection corresponding to eigenvalue -1 of the $SL(2, \mathbb{R})$ part of the affine transformation

$$U : (x, y) \mapsto (-cx + (c-1)y + 1, -(c+1)x + cy + 1)$$

is the slope one direction. It also has the property that $U \circ T_c \circ U^{-1} = T_c^{-1}$, which can be used to show that U swaps P_i with P_{1-i} . Therefore segment $\overline{P_{1-i}P_i}$ always has slope one. The n -th slope one cylinder is formed by considering the union of trapezoid obtained by taking the convex hull of the points P_n, P_{n+1}, P_{1-n} and P_{-n} and the same trapezoid rotated by π inside Q_c^- . Now we will show that the moduli of these cylinders are all equal. The circumference and height of the n -th cylinder in this direction is given below.

$$C_n = \sqrt{2}(x_n - x_{1-n} + x_{n+1} - x_{-n})$$

$$H_n = \frac{\sqrt{2}}{2}(x_{n+1} - x_n, y_{n+1} - y_n) \cdot (-1, 1)$$

Let $P_n = (\hat{x}, \hat{y})$. Then $P_{n+1} = (c\hat{x} + (c-1)\hat{y} + 1, (c+1)\hat{x} + c\hat{y} + 1)$, $P_{1-n} = ((-c)\hat{x} + (c-1)\hat{y} + 1, (-c-1)\hat{x} + c\hat{y} + 1)$ and $P_{-n} = (-\hat{x}, \hat{y})$. We have

$$C_n = \sqrt{2}(2c+2)\hat{x} \quad \text{and} \quad H_n = \sqrt{2}\hat{x}.$$

The modulus of each cylinder is $\frac{1}{2c+2}$. Thus by proposition 4, \hat{E}_c lies in the affine automorphism group. Again, we leave it to the reader to check that the derivative of \hat{E}_c must be E_c . \square

3.2. A classification of saddle connections. In this subsection, we will classify the directions in S_c where saddle connections can appear. We begin with S_1 .

Integers p and q are *relatively prime*, if $n, \frac{p}{n}, \frac{q}{n} \in \mathbb{Z}$ implies $n = \pm 1$. We use the notation $\frac{p}{q} \equiv \frac{r}{s} \pmod{2}$ to say that once the fractions are reduced to $\frac{p'}{q'}$ and $\frac{r'}{s'}$ so that numerator and denominator are relatively

prime, we have $p' \equiv r' \pmod{2}$ and $q' \equiv s' \pmod{2}$. We use $\frac{p}{q} \not\equiv \frac{r}{s} \pmod{2}$ to denote the negation of this statement.

Proposition 6 (Saddle connections of S_1). *Saddle connections $\sigma \subset S_1$ must have integral holonomy $hol_1(\sigma) \in \mathbb{Z}^2$. A direction contains saddle connections if and only if it has rational slope, $\frac{p}{q}$, with $\frac{p}{q} \not\equiv \frac{1}{0} \pmod{2}$.*

Proof. The holonomy of a saddle connection must be integral, because the surface S_1 was built from two (infinite) polygons with integer vertices. The action of the Veech group $\Gamma(S_1)$, the congruence two subgroup of $SL^\pm(2, \mathbb{Z})$ preserves the collection of vectors

$$RP = \{(p, q) \in \mathbb{Z}^2 \setminus \{(0, 0)\} \mid p \text{ and } q \text{ are relatively prime}\}.$$

Indeed the orbits of $(0, 1)$, $(1, 1)$, and $(1, 0)$ under $\Gamma(S_1)$ are disjoint and cover RP . Thus, up to the affine automorphism group, the geodesic flow in a direction of rational slope looks like the geodesic flow in the horizontal, slope one, or vertical directions. All these directions other than the vertical one contain saddle connections. Therefore, rational directions contain saddle connections unless they are in the orbit of the vertical direction under $\Gamma(S_1)$. \square

In order to make a similar statement for S_c , we will need to describe the directions that contain saddle connections. We will find it useful to note that there is a natural bijective correspondence between directions in the plane modulo rotation by π , and the boundary of the hyperbolic plane $\partial\mathbb{H}^2$. This can be seen group theoretically. Directions in the plane correspond to $S^1 = SL(2, \mathbb{R})/\mathbb{H}$ where

$$\mathbb{H} = \{G \in SL(2, \mathbb{R}) \mid G\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} \lambda \\ 0 \end{bmatrix} \text{ for some } \lambda > 0\}.$$

Both directions mod rotation by π and the boundary of the hyperbolic plane correspond to the real projective line, $\mathbb{RP}^1 = SL(2, \mathbb{R})/\mathbb{H}^\pm$, where

$$\mathbb{H}^\pm = \{G \in SL(2, \mathbb{R}) \mid G\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} \lambda \\ 0 \end{bmatrix} \text{ for some } \lambda \neq 0\}.$$

Now consider the left action of the groups \mathcal{G}_c^\pm on \mathbb{RP}^1 . There are known natural semiconjugacies between the actions of \mathcal{G}_c^\pm for $c > 1$ on \mathbb{RP}^1 and the action of \mathcal{G}_1^\pm . That is, for all $c > 1$ there is a continuous surjective map $\varphi_c : \mathbb{RP}^1 \rightarrow \mathbb{RP}^1$ of degree one so that the following

diagram commutes for all $G \in \mathcal{G}^\pm$

$$(4) \quad \begin{array}{ccc} \mathbb{RP}^2 & \xrightarrow{G_c} & \mathbb{RP}^2 \\ \downarrow \varphi_c & & \downarrow \varphi_c \\ \mathbb{RP}^2 & \xrightarrow{G_1} & \mathbb{RP}^2 \end{array}$$

It will also be important that ϕ_c *weakly preserves orientation*. That is, if $(\mathbf{a}, \mathbf{b}, \mathbf{c}) \in (\mathbb{RP}^1)^3$ is a positively oriented triple, then $(\phi_c(\mathbf{a}), \phi_c(\mathbf{b}), \phi_c(\mathbf{c}))$ is not a negatively oriented triple.

We may describe the map $\varphi_c : \mathbb{RP}^1 \rightarrow \mathbb{RP}^1$ geometrically as follows. Recall, that for $c > 1$, the fundamental domain for the action of \mathcal{G}_c^\pm on \mathbb{H}^2 is a triangle Δ_c with two ideal vertices and one ultra-ideal vertex. For $c = 1$, the fundamental domain of \mathcal{G}_1^\pm is a triangle Δ_1 with three ideal vertices. See the right and center sides of figure 3 respectively. We may choose a homeomorphism between the closures in $\overline{\mathbb{H}^2}$, $f_c : \overline{\Delta}_c \rightarrow \overline{\Delta}_1$ which collapses the ultra-ideal vertex to an ideal vertex, and sends the edges to edges. By propagating this homeomorphism around by the groups \mathcal{G}_c^\pm and \mathcal{G}_1^\pm we can find a unique homeomorphism from $\hat{f}_c : \overline{\mathbb{H}^2} \rightarrow \overline{\mathbb{H}^2}$ so that $G_1 \circ f_c = \hat{f}_c \circ G_c|_{\Delta_c}$ for all $G \in \mathcal{G}^\pm$. The desired map φ_c is the restriction of the map \hat{f}_c to $\mathbb{RP}^2 = \partial\mathbb{H}^2$.

The map φ_c is reminiscent of the famous devil's staircase, a continuous surjective map $[0, 1] \rightarrow [0, 1]$ which contracts intervals in the compliment of a Cantor set to points. Indeed, the limit set Λ_c of the group \mathcal{G}_c^\pm is a \mathcal{G}_c^\pm invariant Cantor set, and the connected components of the domain of discontinuity, $\mathbb{RP}^1 \setminus \Lambda_c$, are contracted to points by φ_c .

Recall, the surfaces S_c for $c \geq 1$ are canonically homeomorphic up to isotopy fixing the singular points. We let $h_c : S_c \rightarrow S_1$ be such a homeomorphism. We will see that the saddle connections in S_c and in S_1 are topologically the same. We will now make this notion rigorous. We will say two paths γ and γ' in a translation surface are *homotopic* (relative to their endpoints) if they have the same endpoints and there is a homotopy from γ to γ' which fixes those endpoints. We do not allow this homotopy to pass through singular points. We use $[\gamma]$ to denote the equivalence class of all paths homotopic to γ .

Theorem 7 (Classification of saddle connections). *There is a saddle connection in direction θ of S_c for $c > 1$ if and only if the direction $\varphi_c(\theta)$ contains saddle connections in S_1 . Equivalently, θ contains saddle connections if and only if θ is an image of the horizontal or slope one direction under an element of $\mathcal{G}_c^\pm = \langle -I_c, A_c, D_c, E_c \rangle$. Furthermore, the collection of homotopy classes are identical in S_c and S_1 .*

That is, for all saddle connections $\sigma \subset S_c$ there is a saddle connection $\sigma' \in h_c([\sigma])$ in S_1 , and for all saddle connections $\sigma' \subset S_1$ there is a saddle connection $\sigma \in h_c^{-1}([\sigma'])$ in S_c .

We will prove this theorem by first proving a more abstract lemma. Then we will demonstrate that S_c and S_1 satisfy the conditions of the lemma.

In the statement of the lemma, we use the concept of the holonomy of a saddle connection. Given any oriented path $\gamma : [0, 1] \rightarrow S$ in a translation surface which avoids the singularities on $(0, 1)$, there is a canonical development of γ into the plane, $dev(\gamma) : [0, 1] \rightarrow \mathbb{R}^2$, which is a path in the plane well defined up to translation. The holonomy vector $hol(\gamma)$ is obtained by subtracting the endpoint of $dev(\gamma)$ from its starting point. The quantity $hol(\gamma)$ is homotopy invariant. The notions of holonomy and the developing map are common in the world of (G, X) structures; see section 3.4 of [Thu97], for instance.

The wedge product between two vectors in \mathbb{R}^2 is given by

$$(5) \quad (a, b) \wedge (c, d) = ad - bc.$$

This is also the signed area of the parallelogram formed by the two vectors.

The function $sign : \mathbb{R} \rightarrow \{-1, 0, 1\}$ assigns one to positive numbers, zero to zero, and -1 to negative numbers.

Lemma 8. *Let $h : S \rightarrow T$ be a homeomorphism between translation surfaces satisfying the following statements.*

- (1) *S admits a triangulation by saddle connections.*
- (2) *For every saddle connection $\sigma \subset S$ the homotopy class $h([\sigma])$ contains a saddle connection of T .*
- (3) *Every pair of saddle connections $\sigma_1, \sigma_2 \subset S$ satisfies*

$$sign(hol(\sigma_1) \wedge hol(\sigma_2)) = sign(hol(h(\sigma_1)) \wedge hol(h(\sigma_2))).$$

Then, for every saddle connection $\sigma \subset T$, the homotopy class $h^{-1}([\sigma])$ contains a saddle connection of S .

Proof. Let \mathcal{T}_S be the triangulation of S by saddle connections given to us by item 1. By item 2, we can straighten $h(\mathcal{T}_S)$ to a triangulation \mathcal{T}_T of T by saddle connections.

We define the *complexity* of a saddle connection $\sigma \subset T$ relative to the triangulation \mathcal{T}_T to be the number of times σ crosses a saddle connection in \mathcal{T}_T . We assign the saddle connections in \mathcal{T}_T complexity zero. Supposing the conclusion of the lemma is false, there exists at least one saddle connection $\sigma \subset T$ so that $h^{-1}([\sigma])$ contains no saddle connection of S . We may choose such a saddle connection $\sigma \subset T$ so

that it has minimal complexity with respect to \mathcal{T}_T . By the remarks above this minimal complexity must be at least one.

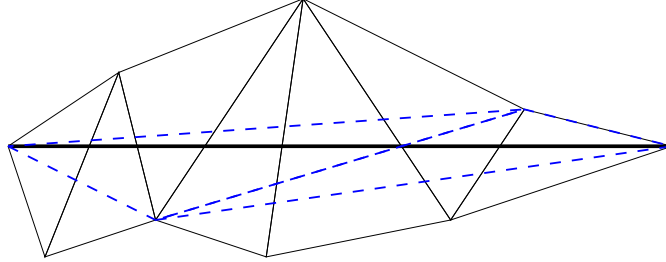


FIGURE 6. The saddle connection σ is developed into the plane along with the triangles in \mathcal{T}_T that it intersects.

Now, we will reduce to a case similar to the complexity one case. We may rotate the surface T so that σ is horizontal and develop the triangles σ intersects into the plane as a chain of triangles. See figure 6. Now choose vertices v_+ and v_- below and above σ which lie closest to σ . The convex hull of $\{\sigma, v_+, v_-\}$ is a quadrilateral Q which is contained in the developed chain of triangles. The boundary of Q consists of four saddle connections ν_1, \dots, ν_4 with complexity relative to \mathcal{T}_T less than that of σ . Because we assumed σ had minimal complexity, there are saddle connections $\nu'_1, \dots, \nu'_4 \subset S$ in the homotopy classes $h^{-1}([\nu_1]), \dots, h^{-1}([\nu_4])$ respectively. Finally, because of item 3, the saddle connections ν'_1, \dots, ν'_4 in S must also form a strictly convex quadrilateral Q' . The quadrilateral Q' must then have diagonals, one of which lies in the homotopy class $h^{-1}([\sigma])$. See figure 7. \square

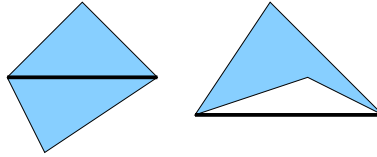


FIGURE 7. To destroy a diagonal of a quadrilateral, the quadrilateral must be made non-convex. This violates property 3 of lemma 8.

The following proposition implies the classification of saddle connections, theorem 7.

Proposition 9. *The canonical homeomorphism $h_c^{-1} : S_1 \rightarrow S_c$ satisfies the conditions of lemma 8.*

Proof. Item 1 is trivial. We leave it to the reader to triangulate S_1 .

Item 2 follows from proposition 6 together with work done in subsection 3.1. By proposition 6 all saddle connections of S_1 are the images of saddle connections in the horizontal and slope one directions under \mathcal{G}_1^\pm . By work done in subsection 3.1, we know that for each saddle connection σ in the horizontal and slope one directions that appears in S_1 , there is a saddle connection $\sigma' \in h_c^{-1}([\sigma])$ in S_c . Furthermore by lemma 5, the group action of $\langle \widehat{-I}_c, \widehat{A}_c, \widehat{D}_c, \widehat{E}_c \rangle \subset \text{Aut}(S_c)$ on S_c is the same as the group action of $\langle \widehat{-I}_1, \widehat{A}_1, \widehat{D}_1, \widehat{E}_1 \rangle \subset \text{Aut}(S_1)$ on S_1 up to h_c and isotopies preserving the singularities. Therefore, for every saddle connection $\sigma \subset S_1$, there is a saddle connection $\sigma' \in h_c^{-1}([\sigma])$.

Now we show item 3 holds. Let σ and σ' be saddle connections in S_1 . Let θ_0, θ_1 be the horizontal and slope one directions in the circle S^1 , respectively. Then we can choose $\mathbf{v}, \mathbf{v}' \in \{\theta_0, \theta_1\}$ and $G_1, G'_1 \in \mathcal{G}_1^\pm$ such that $G_1(\mathbf{v}) = \text{hol}_1(\sigma)$ and $G'_1(\mathbf{v}') = \text{hol}_1(\sigma')$. It follows that the corresponding elements $G_c, G'_c \in \mathcal{G}_c^\pm$ satisfy $G_c(\mathbf{v}) = \text{hol}_c \circ h_c^{-1}(\sigma)$ and $G'_c(\mathbf{v}') = \text{hol}_c \circ h_c^{-1}(\sigma')$. We must prove that

$$\text{sign}(G_1(\mathbf{v}) \wedge G'_1(\mathbf{v}')) = \text{sign}(G_c(\mathbf{v}) \wedge G'_c(\mathbf{v}')).$$

This follows essentially from the description of the map φ_c . Let $\tilde{\varphi}_c : S^1 \mapsto S^1$ be a lift of $\varphi_c : \mathbb{RP}^1 \rightarrow \mathbb{RP}^1$. This lift is unique if we assume the rightward direction is preserved, $\tilde{\varphi}_c(1, 0) = (1, 0)$. The map $\tilde{\varphi}_c$ satisfies the commutative diagram 4 with \mathbb{RP}^1 replaced by S^1 . Using this diagram we may rewrite the above equation as

$$\text{sign}(\tilde{\varphi}_c \circ G_c(\mathbf{v}) \wedge \tilde{\varphi}_c \circ G'_c(\mathbf{v}')) = \text{sign}(G_c(\mathbf{v}) \wedge G'_c(\mathbf{v}')).$$

Since, φ_c weakly preserves orientation, it follows that $\tilde{\varphi}_c$ weakly preserves sign of \wedge . That is, if \mathbf{u} and \mathbf{w} are two unit vectors, then

$$\text{sign}(\tilde{\varphi}_c(\mathbf{u}) \wedge \tilde{\varphi}_c(\mathbf{w})) = 0 \quad \text{or} \quad \text{sign}(\tilde{\varphi}_c(\mathbf{u}) \wedge \tilde{\varphi}_c(\mathbf{w})) = \mathbf{u} \wedge \mathbf{w}.$$

Thus, our only fear is that $G_c(\mathbf{v})$ and $G'_c(\mathbf{v}')$ are distinct in \mathbb{RP}^1 , but their images $\varphi_c \circ G_c(\mathbf{v})$ and $\varphi_c \circ G'_c(\mathbf{v}')$ were not distinct. But directions containing saddle connections have parabolics that preserve them. So there are parabolics $P_c, P'_c \in \mathcal{G}_c^\pm$ which fix $G_c(\mathbf{v})$ and $G'_c(\mathbf{v}')$ in \mathbb{RP}^1 , respectively. The corresponding parabolics $P_1, P'_1 \in \mathcal{G}_1^\pm$ fix the directions $\varphi_c \circ G_c(\mathbf{v})$ and $\varphi_c \circ G'_c(\mathbf{v}')$ in \mathbb{RP}^1 . Now, by the commutative diagram 4, the points $\varphi_c \circ G_c(\mathbf{v})$ and $\varphi_c \circ G'_c(\mathbf{v}')$ in \mathbb{RP}^1 have unique preimages under φ_c^{-1} . So, $\varphi_c \circ G_c(\mathbf{v})$ and $\varphi_c \circ G'_c(\mathbf{v}')$ must be distinct after all. \square

The next theorem does not contribute to the classification of affine automorphisms, but culminates the discussion in this section. Suppose we have a triangulation \mathcal{T} of a translation surface S . The triangulation can be used to code trajectories under the geodesic flow on S . We code

every trajectory by the sequence of saddle connections in \mathcal{T} that the trajectory crosses. We call this the *orbit-type* of the trajectory.

For some $c_1 \geq 1$ fix a triangulation \mathcal{T}_{c_1} of S_{c_1} . By theorem 7, there are identical triangulations \mathcal{T}_c of every S_c with $c \geq 1$. The saddle connections in each \mathcal{T}_c are homotopic to the corresponding saddle connection in \mathcal{T}_{c_1} up to the canonical homeomorphism $S_c \rightarrow S_{c_1}$.

Theorem 10. *Let \mathcal{T}_c be a triangulation of S_c for each $c \geq 1$, as described above. Then, given any $c_1, c_2 \geq 1$ and any finite or infinite trajectory γ_1 in S_{c_1} , there is a trajectory γ_2 in S_{c_2} with the same orbit type.*

Again, this theorem is implied by a more general statement.

Lemma 11. *Suppose $h : S \rightarrow S'$ is a homeomorphism between translation surfaces satisfying the three statements of lemma 8. Let \mathcal{T} be a triangulation of S and \mathcal{T}' be the corresponding triangulation of S' . Then, given any finite or infinite trajectory γ in S , there is a trajectory γ' in S' with the same orbit type.*

Proof. We will prove the statement for finite trajectories. The statement for infinite trajectories follows by looking at longer and longer finite subtrajectories.

Let γ be a finite trajectory in S . To derive a contradiction, it would have to be that γ crosses at least two saddle connections in \mathcal{T} . Develop γ into the plane, along with the saddle connections it crosses. The sequence of saddle connections in the orbit-type develop to a sequence of segments s_i in the plane. We define Λ to be the space of all lines in \mathbb{R}^2 that cross through the interiors of each segment s_i . The *slalom hull* of the sequence $\langle s_i \rangle$ is

$$SH = \bigcup_{\ell \in \Lambda} \ell.$$

Clearly this slalom hull is non-empty, because the line containing the developed image of γ lies in Λ . The boundary of the SH consists of four rays and a finite set of segments that pull back to connections. Orient SH so that it is nearly horizontal as in the figure 8. Note that $\mathbb{R}^2 \setminus SH$ consists of two convex components. We call the finite segments in the top component of ∂SH the *top chain*. Similarly, the segments in the bottom component will be called the bottom chain. Further, note there are *diagonals*, which are parallel to the infinite rays in ∂SH . They are formed by connecting the left-most vertex of the top chain to the right-most vertex of the bottom chain, and vice versa. We call the union of the top chain, the bottom chain, and the diagonals the *saddle chain*, and note that they form a loop.

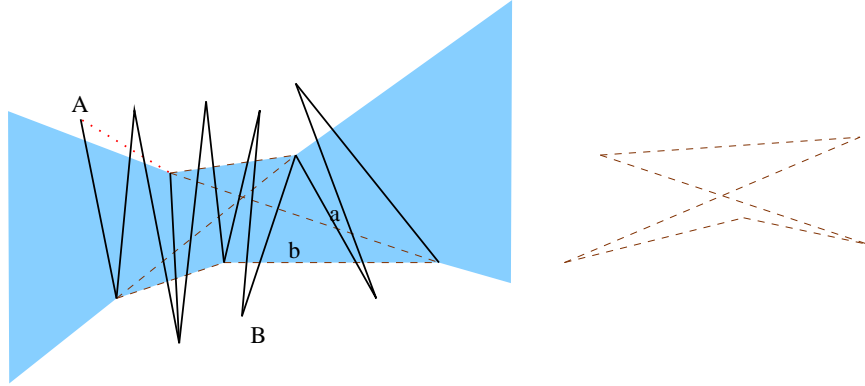


FIGURE 8. On the left, The slalom hull of a sequence of solid segments is rendered as the shaded polygon. The slalom chain is the sequence of dotted lines. The right shows a deformation of the slalom chain. No deformation of the slalom chain which preserves the cyclic ordering of edges in \mathbb{RP}^2 can destroy the slalom hull.

Each c_i in the saddle chain of S pulls back to a saddle connection σ_i of S . Let σ'_i be the corresponding saddle connections in S' . By item 3 of lemma 8, the directions of saddle connections σ'_i must be in the same cyclic order as the directions for σ_i . We may develop the saddle connections σ'_i into the plane to obtain the chain of segments c'_i . This chain of segments has essentially the same combinatorics. Consider the sequence of segments given by the diagonal running from top left to bottom right, and then moving across the top chain from left to right, and ending with the other diagonal. Each segment must be rotated slightly counterclockwise to reach the subsequent one. Thus, the region bounded by the top chain, by the ray leaving the top left vertex in the direction of the diagonal, and the ray leaving the top right vertex in the direction of the other diagonal must bound a convex set. Similarly, there is another natural convex set bounded by the lower chain and some rays. The convex sets may not intersect, because they are guaranteed to lie in two opposite quadrants of the division of the plane by lines through the diagonals. Thus, we have an analogous set of lines Λ' which pass through the diagonals and not the lower and upper chains. The slalom hull SH' given by the same formula is non-empty.

We claim that SH' is the slalom hull for the segments $\{s'_i\}$. This is equivalent to saying that no endpoints of s'_i lie within SH' , which will imply our theorem. Some of the locations of endpoints are determined, because they are endpoints of segments in the chain $\{c'_i\}$.

Suppose $B \in \mathbb{R}^2 \setminus SH$ is an endpoint of a segment s_i in the development of S which crosses one of the edges in the slalom hull, $b \in \partial SH$. See figure 8. Let B' be the corresponding endpoint of the segment s'_i in the development from S' . Then B' cannot be in SH' , lest it destroy the segment c'_i , which must pull back to a saddle connection in S' .

The more difficult case is when $A \in \mathbb{R}^2 \setminus SH$ is an endpoint of a segment s_i in the development S which crosses one of the rays in ∂SH . Without loss of generality, assume the ray crossed is the upper left ray in the development. Let a be the diagonal element of the slalom chain that is parallel to this ray, and a' the corresponding element in the development of S' . We consider the canonical homotopy class of S , with the cone singularities removed, of paths joining the pull back of the top left vertex in ∂SH to the pull back A . This homotopy class should contain the path which develops to follow the ray of ∂SH leaving the top left vertex until it hits the segment s_i and then follows s_i to A . Let d be the infimum of the lengths of paths in this homotopy class. There is a limiting path $p \subset S$ that may pass through singularities, that achieves this infimum. This path consists of a sequence of saddle connections, and turns only rightward in total angle less than π . The initial segment of the developed image of p must immediately leave SH , hence its orientation when compared to a is determined. Let p' be the corresponding chain of saddle connections in S' , which must likewise turn rightward by total angle less than π . The initial segment of the developed image of p' must immediately leave SH' , because its orientation when compared to a' must match that of the case in S . Moreover, since p' only bends rightward, the path cannot return to SH' . This concludes the argument that SH' is the slalom hull for the segments $\{s'_i\}$. We know this contains lines, any of which contain a segment which pulls back to a trajectory γ' in S' with the same orbit type as γ in S . \square

3.3. No other affine automorphisms. The last step to the proof of theorems 1 and 2 is to demonstrate that all affine automorphisms of the surface lie in the group generated by the elements we listed.

Lemma 12. *All affine automorphisms of the surface are contained in the group generated by $\widehat{-I}_c$, \widehat{A}_c , \widehat{D}_c , and \widehat{E}_c .*

Proof. Let us suppose that for some $c \geq 1$ there is an $M \in GL(2, \mathbb{R})$ in the Veech group $\Gamma(S_c)$ and a corresponding element \widehat{M} in the affine automorphism group $Aut(S_c)$. We will prove that \widehat{M} lies in the group generated by the four elements $\widehat{-I}_c$, \widehat{A}_c , \widehat{D}_c , and \widehat{E}_c .

Let $\theta = (1, 0) \in S^1$ be the horizontal direction. We know that the image $M(\theta)$ must contain saddle connections of S_c . Further more the horizontal and slope one directions can be distinguished, since the smallest area maximal cylinder in the horizontal direction has two cone singularities in its boundary, while the smallest area maximal cylinder in the slope one direction has four cone singularities in its boundary. Thus, by theorem 7, there must be an element $N_c \in \mathcal{G}_c^\pm$ satisfying $M(\theta) = N_c(\theta)$. It follows that $N_c^{-1} \circ M$ preserves the horizontal direction.

There must be a corresponding element $\hat{N}_c^{-1} \circ \hat{M} \in \text{Aut}(S_c)$ with derivative $N_c^{-1} \circ M$. The automorphism must fix the decomposition into horizontal cylinders, and fix each cylinder in the decomposition (because the cylinders have distinct areas). The smallest area horizontal cylinder is isometric in each S_c . It is built from two triangles, the convex hull of $(0, 0)$, $(1, 1)$, and $(-1, 1)$ and the same triangle rotated by π , with diagonal sides of the first glued to the diagonal sides of the second by translation. $\hat{N}_c^{-1} \circ \hat{M} \in \text{Aut}(S_c)$ must preserve this cylinder and permute the pair of cone singularities in the boundary. Therefore

$$N_c^{-1} \circ M = \begin{bmatrix} 1 & 2n \\ 0 & 1 \end{bmatrix} = D_c^n \quad \text{or} \quad N_c^{-1} \circ M = \begin{bmatrix} 1 & -2n \\ 0 & -1 \end{bmatrix} = -I \circ A_c \circ D_c^n$$

for some $n \in \mathbb{Z}$. Therefore, $M = N_c \circ D_c^n$ or $M = N_c \circ -I \circ A_c \circ D_c^n$, all of which lie in \mathcal{G}_c^\pm . Therefore, $\Gamma(S_c) \subset \mathcal{G}_c^\pm$.

Now, we have determined all of the affine automorphism group unless there is something in the kernel of the derivative map $\mathbf{D} : \text{Aut}(S_c) \rightarrow \Gamma(S_c)$. Suppose there was a non-trivial element \hat{M} in the kernel. Then it would have to fix the smallest horizontal cylinder. Because there is only one cone point in the top boundary component, \hat{M} must fix every point in the smallest cylinder. Then, it must fix every point in S_c , since S_c is path connected. \square

4. RECURRENCE OF THE GEODESIC FLOW ON S_1

In this section, we will demonstrate the following.

Theorem 13 (Recurrence). *The geodesic flow in irrational directions on the surface S_1 is recurrent.*

A nice geometric consequence is that there are no strips (subsets isometric to $\mathbb{R}_{>0} \times (0, a)$ for $a > 0$) in irrational directions. Earlier work also tells us that there are no closed trajectories and no saddle connections in irrational directions.

The following corollary summarizes the situation in all directions. The notation $\frac{p}{q} \equiv \frac{p'}{q'} \pmod{2}$ means that when both fractions are reduced, $p \equiv p' \pmod{2}$ and $q \equiv q' \pmod{2}$.

Corollary 14 (A trichotomy). *Let $m \in \mathbb{R} \cup \{\infty\}$ be a slope. Then the geodesic flow in direction m on S_1 satisfies*

- *If $m = \frac{p}{q}$ is rational and $\frac{p}{q} \equiv \frac{0}{1} \pmod{2}$ or $\frac{p}{q} \equiv \frac{1}{1} \pmod{2}$ then m is a completely periodic direction. That is, there is a decomposition into cylinders in that direction and every trajectory is closed or a saddle connection.*
- *If $m = \frac{p}{q}$ is rational and $\frac{p}{q} \equiv \frac{1}{0} \pmod{2}$, then there is a decomposition of the surface into infinite strips isometric to $I \times \mathbb{R}$ with $I \subset \mathbb{R}$ an open interval. Every trajectory is singular in at most one direction (forward or backward). Any trajectory which is not singular in forward (resp. backward) time eventually leaves every compact set and never returns.*
- *If m is irrational, then the geodesic flow in this direction is recurrent. However, there are no closed trajectories or saddle connections in this direction.*

The case for rational directions follows from earlier work. Every rational direction is the image under $\Gamma(S_1)$, the congruence two subgroup of $SL(2, \mathbb{Z})$, of the horizontal, vertical, or slope one directions.

Now we will discuss the proof of theorem 13. Given an irrational direction θ on the surface, we get a foliation of the surface by leaves which travel in the direction θ . There is a natural transverse measure μ_θ on the leaves, which measures the Euclidean width of leaves a transversal crosses. For each irrational θ we will construct a nested sequence of subsurfaces $D_1 \subset D_2 \subset D_3 \subset \dots$ satisfying the following lemma.

Lemma 15 (Nested exhaustion). *Suppose S is a translation surface and θ is a direction. Suppose also there exists a nested sequence of finite area compact subsurfaces $D_1 \subset D_2 \subset D_3 \subset \dots$ satisfying the following two conditions.*

- (1) $S = \bigcup_{n \in \mathbb{Z}} D_n$.
- (2) $\liminf_{n \rightarrow \infty} \mu_\theta(\partial D_n) = 0$.

Then the geodesic flow in direction θ is recurrent.

This lemma is not a new idea. It is used in work of Gutkin and Troubetzkoy, where it is used to prove that the geodesic flow in right triangles is directionally recurrent [GT96].

Proof of the nested exhaustion lemma. Suppose the conditions of the lemma are satisfied. Fix a finite transversal I to the foliation in the

direction θ , we will show that there is a first return map under the flow in direction θ which is well defined almost everywhere. We can find an N so that $I \subset D_N$. Given any $n \geq N$ we define a transverse measure preserving map on a full measure subset $J \subset I$ of the form $\phi_n : J \rightarrow I \cup \partial D_n$. Given a point $x \in I$, simply flow in direction θ until you hit either I or ∂D_n , the image $\phi_n(x)$ is that point that you first hit. This is not defined for every point, because the trajectory of a point might never hit I or ∂D_n . But, because D_n is finite area, it is well defined on a subset of full measure. Using the fact that ϕ_n is measure preserving we see

$$\mu_\theta(\phi_n^{-1}(I)) > \mu_\theta(I) - \mu_\theta(\partial D_n).$$

Note, by condition 2 of the lemma, there is a subsequence of times n so that the right side converging to $\mu_\theta(I)$ as $n \rightarrow \infty$. Furthermore, if $m > n$ and x is a point so that $\phi_n(x) \in I$ then $\phi_m(x) = \phi_n(x)$. Thus, it makes sense to define a limiting function $\phi_\infty : I \rightarrow I$ where $\phi_\infty(x) = y$ if there is an n with $\phi_n(x) = y$. Because of the above argument, this function is defined on a subset of full measure and realizes the first return map to I . Thus the flow in direction θ is recurrent. \square

In the next two subsections we will prove that this lemma applies to our situation.

4.1. Subsurfaces built from cylinders. The maximal cylinders in a completely periodic direction of S_1 have a canonical numbering by \mathbb{N} in order of increasing areas. We identify a rational direction using a reduced fractional slope $\frac{p}{q} \in \mathbb{RP}^1$, representing the direction the vector $(q, p) \in \mathbb{R}^2$ points. We allow this to include the fraction $\infty = \frac{1}{0}$. We will use $\mathcal{C}_{q,p}^n$ to denote the n -th cylinder in the direction $\frac{q}{p}$.

Let $\mathcal{D}_{q,p}^n$ denote the closure of the union of the cylinders $\mathcal{C}_{q,p}^1, \mathcal{C}_{q,p}^2, \dots, \mathcal{C}_{q,p}^n$. Note that by definition we have $\mathcal{D}_{q,p}^n \subset \mathcal{D}_{q,p}^{n+1}$. The boundaries of the surfaces $\mathcal{D}_{q,p}^n$ are described by the following proposition.

Proposition 16. *Let $\frac{p}{q} \not\equiv \frac{1}{0} \pmod{2}$. The boundary of the surface $\mathcal{D}_{q,p}^n$ consists of two saddle connections. When $\frac{p}{q} \equiv \frac{0}{1} \pmod{2}$, they each have holonomy $(2nq, 2np)$. When $\frac{p}{q} \equiv \frac{1}{1} \pmod{2}$, they each have holonomy $((2n+1)q, (2n+1)p)$.*

Proof. The statement is invariant under the affine automorphism group. Thus the statement only needs to be checked in the horizontal and slope one directions. We leave it to the reader to check these two cases. \square

The *Farey graph* \mathbf{F} is a graph whose vertices are the rational points of \mathbb{RP}^1 (i.e. the vertices are points in \mathbb{QP}^1). Two rational points $\frac{p}{q}$ and

$\frac{r}{s}$ are the vertices of an edge of the Farey graph if $ps - rq = \pm 1$. Geometrically, \mathbf{F} may be realized as an $SL^\pm(2, \mathbb{Z})$ invariant triangulation of the hyperbolic plane by ideal triangles.

The following proposition summarizes the containment properties of the subsurfaces of the form $\mathcal{D}_{q,p}^n$.

Proposition 17. *Assume the fractions $\frac{p}{q} \equiv \frac{0}{1} \pmod{2}$ and $\frac{r}{s} \equiv \frac{1}{1} \pmod{2}$ are adjacent in the Farey graph. Then for all $n \in \mathbb{N}$,*

$$\mathcal{D}_{q,p}^n \subset \mathcal{D}_{s,r}^n \quad \text{and} \quad \mathcal{D}_{s,r}^n \subset \mathcal{D}_{q,p}^{n+1}$$

Proof. By theorem 1, the Veech group of S_1 is the congruence two subgroup of $SL(2, \mathbb{Z})$. This group acts transitively on edges of the form mentioned in the proposition. The statement is affine automorphism invariant, so it is sufficient to prove it for the case of $\frac{p}{q} = \frac{0}{1}$ and $\frac{r}{s} = \frac{1}{1}$. This case is illustrated by figure 4. \square

This proposition encapsulates all we know about the containment properties of the subsurfaces $\mathcal{D}_{q,p}^n$. So, the Farey graph contains many edges that will be of no use to us. We define the subgraph $\mathbf{G} \subset \mathbf{F}$ to be the graph obtained by removing vertices corresponding to rationals equivalent to $\frac{1}{0}$ modulo 2. All edges which connect to a vertex equivalent to $\frac{1}{0}$ modulo 2 must also be removed. The graph \mathbf{G} is depicted in figure 9.

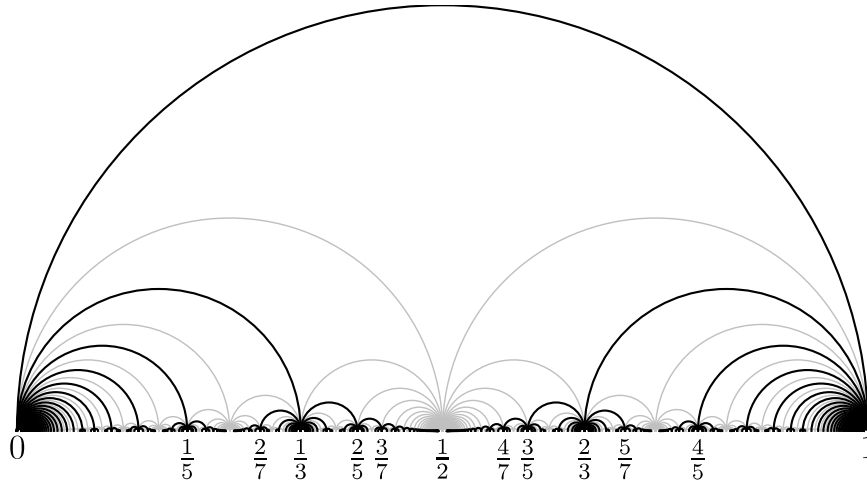


FIGURE 9. The dark lines illustrate the portion of \mathbf{G} within $[0, 1]$. The Farey graph includes the lighter lines. Both graphs are invariant under integer translations.

The graph \mathbf{G} is a tree. In fact, it is the tree with countably many edges meeting at every vertex. Thus, paths between two vertices which do not backtrack over themselves are unique. Furthermore, if $\frac{p_1}{q_1} \in \mathbf{G}$ is a vertex and $\theta \in \mathbb{R} \subset \mathbb{RP}^1$ is irrational, there is a unique non-backtracking path so that the vertices crossed by the path $\frac{p_1}{q_1}, \frac{p_2}{q_2}, \dots$ converge to θ . When $\frac{p_1}{q_1} = n$ is the closest integer to θ , we call this sequence the \mathbf{G} -sequence.

There is a natural coding of the \mathbf{G} -sequence converging to θ of the form $(\frac{p_1}{q_1}, \frac{p_2}{q_2}; k_2, k_3, \dots)$. $\frac{p_1}{q_1}$ and $\frac{p_2}{q_2}$ are the first two rationals in the \mathbf{G} -sequence. Each $k_i \in \mathbb{Z} \setminus 0$. For $i \geq 2$, assume the $i-1$ -st and i -th fractions in the \mathbf{G} -sequence are given by $\frac{p_{i-1}}{q_{i-1}}$ and $\frac{p_i}{q_i}$. Then all the rationals which are adjacent in the Farey graph to $\frac{p_i}{q_i}$ are of the form $\frac{p_{i-1}+jp_i}{q_{i-1}+jq_i}$. Thus, there is a j so that the $i+1$ -st rational is given by $\frac{p_{i+1}}{q_{i+1}} = \frac{p_{i-1}+jp_i}{q_{i-1}+jq_i}$. In order for this fraction not to be congruent to $\frac{1}{0}$ modulo two, the number j must be even. To avoid backtracking, $j \neq 0$. So, we set add the number $k_i = \frac{j}{2}$ to our coding. This number satisfies

$$(6) \quad \frac{p_{i+1}}{q_{i+1}} = \frac{p_{i-1} + 2k_i p_i}{q_{i-1} + 2k_i q_i},$$

where we simplify fractions so that the denominator is always positive.

Now we will describe the nested sequence of surfaces used to apply lemma 15. Let $\theta \in \mathbb{R} \subset \mathbb{RP}^1$ be any irrational slope and consider the \mathbf{G} -sequence of θ , $\langle \frac{p_i}{q_i} \rangle$. We consider the sequence of surfaces $\langle D_{q_i, p_i}^i \rangle_{i \in \mathbb{N}}$. By proposition 17 each surface is contained in the subsequent surface. Using proposition 16, we compute that

$$\mu_\theta(\partial D_{q_i, p_i}^i) = \begin{cases} \frac{2i}{\sqrt{1+\theta^2}} |(1, \theta) \wedge (q_i, p_i)| & \text{if } \frac{p_i}{q_i} \equiv \frac{0}{1} \pmod{2} \\ \frac{2i+1}{\sqrt{1+\theta^2}} |(1, \theta) \wedge (q_i, p_i)| & \text{if } \frac{p_i}{q_i} \equiv \frac{1}{1} \pmod{2}. \end{cases}$$

We will find it convenient to notice that

$$(7) \quad \mu_\theta(\partial D_{q_i, p_i}^i) \leq \frac{2i+1}{\sqrt{1+\theta^2}} |(1, \theta) \wedge (q_i, p_i)| = \frac{(2i+1)q_i}{\sqrt{1+\theta^2}} \left| \theta - \frac{p_i}{q_i} \right|$$

In order to apply lemma 15, it is sufficient to show that the lim inf of the right hand side as $i \rightarrow \infty$ is zero. We see this sufficient condition is fulfilled if

$$(8) \quad \liminf_{i \rightarrow \infty} i q_i \left| \theta - \frac{p_i}{q_i} \right| = 0,$$

since there is a constant κ depending only on θ so that $\kappa i q_i > \frac{(2i+1)q_i}{\sqrt{1+\theta^2}}$. Taking $\kappa = \frac{3}{\sqrt{1+\theta^2}}$ suffices. We will prove equation 8 is true in the

next subsection. The following lemma is the main contribution of our knowledge of the \mathbf{G} -sequence.

Lemma 18. *Let $\langle \frac{p_i}{q_i} \rangle$ be the \mathbf{G} -sequence for an irrational θ . Then*

$$\lim_{i \rightarrow \infty} \frac{i}{q_i} = 0.$$

Proof. Consider the recursive definition for the denominator given in equation 6.

$$q_{i+1} = |q_{i-1} + 2k_i q_i|.$$

It can be seen inductively that the sequence of denominators is strictly increasing. Furthermore,

$$(9) \quad q_{i+1} - q_i \geq q_i - q_{i-1}.$$

Equality only happens when $k_i = -1$. We break into two cases.

First, suppose that infinitely many k_i are not equal to -1 . Fix an arbitrary $c > 0$. Then, by the previous paragraph, there is a $N > 0$ such that $q_{N+1} - q_N > c$. Then by equation 9 we know $q_{N+j} > q_N + jc$ for all $j \geq 1$. Consequently,

$$\limsup_{i \rightarrow \infty} \frac{i}{q_i} < \limsup_{j \rightarrow \infty} \frac{N+j}{q_N + jc} = \frac{1}{c}.$$

Since c was arbitrary the conclusion of the lemma must hold.

If the first case does not hold, there is an $N \in \mathbb{N}$ such that $k_i = -1$ for all $i \geq N$. So for $i \geq N$, we obtain the inductive formula

$$q_{i+1} = 2q_i - q_{i-1} \quad \text{and} \quad p_{i+1} = 2p_i - p_{i-1}.$$

It follows that for all $j \geq 0$, we know $q_{N+j} = q_N + j(q_N - q_{N-1})$ and $p_{N+j} = p_N + j(p_N - p_{N-1})$. Then

$$\theta = \lim_{j \rightarrow \infty} \frac{p_{N+j}}{q_{N+j}} = \lim_{j \rightarrow \infty} \frac{p_N + j(p_N - p_{N-1})}{q_N + j(q_N - q_{N-1})} = \frac{p_N - p_{N-1}}{q_N - q_{N-1}}.$$

But this contradicts the assumption that θ is irrational. That is, the coding of the \mathbf{G} -sequence for an irrational can not end in an infinite sequence of -1 s. \square

4.2. Continued fractions. We will now introduce continued fractions. By the notation $[a_0; a_1, a_2, \dots, a_k]$ we mean the continued fraction

$$[a_0; a_1, a_2, \dots, a_k] = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{\ddots + a_{k-1} + \frac{1}{1 + \frac{1}{a_k}}}}}$$

See [Khi97] and [RS92] for introductions to continued fractions.

Associated to an irrational θ is a unique sequence of rational approximates $\frac{s_i}{t_i}$, called *the continued fraction approximates*, a sequence of integers $[a_0; a_1, a_2, \dots]$, which is the *continued fraction expansion*, and a sequence of irrational *remainders* r_0, r_1, \dots . For purposes of the inductive argument, we define $s_{-1} = 1$ and $t_{-1} = 0$. We define a_0 to be the largest integer less than θ , $\frac{s_0}{t_0} = \frac{a_0}{1}$, and $r_0 = \theta$. It will inductively be true that

$$(10) \quad 0 < r_i - a_i < 1.$$

Given definitions of a_i , $\frac{s_i}{t_i}$, and r_i , we define r_{i+1} to be the unique number satisfying

$$r_i = a_i + \frac{1}{r_{i+1}}.$$

By our inductive hypothesis, $r_{i+1} > 1$. We choose a_{i+1} to be the largest integer less than r_{i+1} , so that a_{i+1} and r_{i+1} satisfy the inductive hypothesis, equation 10. Then we define

$$\frac{s_{i+1}}{t_{i+1}} = [a_0; a_1, \dots, a_{i+1}]$$

When expanded, we get the more useful formula

$$\frac{s_{i+1}}{t_{i+1}} = \frac{a_{i+1}s_i + s_{i-1}}{a_{i+1}t_i + t_{i-1}}.$$

It is well known that the continued fraction sequence for θ converges to θ . Moreover, we have the estimate

$$(11) \quad \left| \theta - \frac{s_i}{t_i} \right| < \frac{1}{q_i^2}.$$

Proposition 19. *Infinitely many fractions in the continued fraction sequence for an irrational θ also lie in the \mathbf{G} -sequence of θ .*

Proof. A pair of adjacent continued fraction approximates can be verified to lie on opposite sides of the number θ . Also, they are adjacent in the Farey graph. Choose i large enough so that $t_i > 1$. Then, the

interval with endpoints $\frac{s_i}{t_i}$ and $\frac{s_{i+1}}{t_{i+1}}$ contains θ , but no integers. Let E denote the edge of the Farey graph with endpoints $\frac{s_i}{t_i}$ and $\frac{s_{i+1}}{t_{i+1}}$. Any path through the Farey graph starting at an integer, which limits on θ must pass through E . Therefore, one of the endpoints of E must appear in the \mathbf{G} -sequence. We conclude that at least one of every adjacent pair of continued fraction approximates must be in the \mathbf{G} -sequence. \square

Proof of theorem 13. Following the logic of the previous subsection, to apply lemma 15, it is sufficient to prove equation 8. Let $\frac{p_{i(j)}}{q_{i(j)}}$ denote the subsequence of the \mathbf{G} -sequence, $\frac{p_i}{q_i}$, for θ which are in the continued fraction sequence for θ . This is a subsequence because of the previous proposition. Using the estimate 11 we may simplify the left hand side of equation 8.

$$\liminf_{i \rightarrow \infty} i q_i \left| \theta - \frac{p_i}{q_i} \right| \leq \liminf_{j \rightarrow \infty} i(j) q_{i(j)} \left| \theta - \frac{p_{i(j)}}{q_{i(j)}} \right| \leq \liminf_{j \rightarrow \infty} \frac{i(j)}{q_{i(j)}} = 0$$

The last step follows directly from lemma 18. \square

5. DYNAMICS OF THE AFFINE AUTOMORPHISM GROUP

This section concerns asymptotics of areas of cylinder intersections under the action of the affine automorphism group. This turns out to be roughly equivalent to studying the action of the affine automorphism on homology.

This section culminates with the proof of equation 1 of the introduction, which is restated as theorem 30.

5.1. Intersections of cylinders. Consider two cylinders \mathcal{A} and \mathcal{B} in the surface S_1 , and an element \hat{H} in the affine automorphism group $Aut(S_1)$ acting hyperbolically. Algebraically, acting hyperbolically means the derivative $H = \mathbf{D}(\hat{H})$ in the Veech group has an eigenvalue bigger than one. We are interested in studying the asymptotics of the quantity

$$Area(\hat{H}^n(\mathcal{A}) \cap \mathcal{B})$$

as n tends to infinity.

Let $\gamma_{\mathcal{A}}$ and $\gamma_{\mathcal{B}}$ be core curves of the cylinders \mathcal{A} and \mathcal{B} respectively. Given a closed curve on a translation surface, we can develop it into the plane. This development is canonical up to translations. The translation vector taking the initial point of this development to the termination point is the *holonomy* around the closed curve. Let $hol(\gamma_{\mathcal{A}}), hol(\gamma_{\mathcal{B}}) \in \mathbb{R}^2$ denote the holonomy around the core curves.

The wedge product between two vectors in \mathbb{R}^2 is the usual one described by equation 5. This is also the signed area of the parallelogram formed by the two vectors.

Proposition 20. *Two non-parallel cylinders \mathcal{A} and \mathcal{B} on a translations surface satisfy the equation*

$$\text{Area}(\mathcal{A} \cap \mathcal{B}) = \frac{|\gamma_{\mathcal{A}} \cap \gamma_{\mathcal{B}}| \text{Area}(\mathcal{A}) \text{Area}(\mathcal{B})}{|\text{hol}(\gamma_{\mathcal{A}}) \wedge \text{hol}(\gamma_{\mathcal{B}})|},$$

where $|\gamma_{\mathcal{A}} \cap \gamma_{\mathcal{B}}|$ denotes the absolute value of the algebraic intersection number.

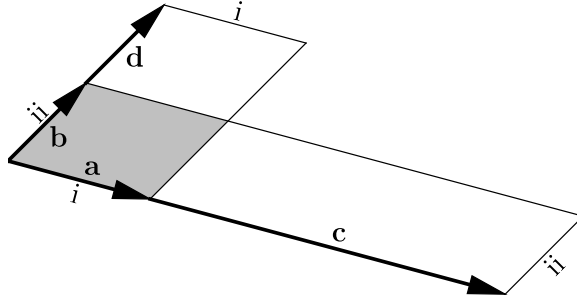


FIGURE 10. Two intersecting cylinders developed into the plane. Roman numerals indicate edge identifications, which reconstruct the cylinders.

Proof. Cylinders on a translation surface intersect in a union of parallelograms, which are isometric and differ only by parallel translation. The number of these parallelograms is the absolute value of the algebraic intersection number between the core curves. Thus, we need to show that the area of one such parallelogram is given by

$$(12) \quad \frac{\text{Area}(\mathcal{A}) \text{Area}(\mathcal{B})}{|\text{hol}(\gamma_{\mathcal{A}}) \wedge \text{hol}(\gamma_{\mathcal{B}})|}.$$

Develop the two cylinders into the plane from an intersection as in figure 10. Define the vectors \mathbf{a} , \mathbf{b} , \mathbf{c} , and \mathbf{d} as in the figure. Then the area of the two cylinders is given by the quantities

$$\text{Area}(\mathcal{A}) = |\mathbf{a} \wedge (\mathbf{b} + \mathbf{d})| \quad \text{and} \quad \text{Area}(\mathcal{B}) = |(\mathbf{a} + \mathbf{c}) \wedge \mathbf{b}|.$$

Since the pair of vectors \mathbf{a} and \mathbf{c} are parallel as are the pair \mathbf{b} and \mathbf{d} , we may write the product of areas as

$$\text{Area}(\mathcal{A}) \text{Area}(\mathcal{B}) = |\mathbf{a} \wedge \mathbf{b}| |(\mathbf{a} + \mathbf{c}) \wedge (\mathbf{b} + \mathbf{d})|.$$

The wedge of the holonomies may be written as

$$|hol(\gamma_{\mathcal{A}}) \wedge hol(\gamma_{\mathcal{B}})| = |(\mathbf{a} + \mathbf{c}) \wedge (\mathbf{b} + \mathbf{d})|.$$

Thus, the quotient given in equation 12 is $|\mathbf{a} \wedge \mathbf{b}|$, the area of the parallelogram formed by the intersection. \square

Let us apply this proposition to our problem. We use the facts that $\hat{H}(hol(\gamma_{\mathcal{A}})) = hol(\hat{H}(\gamma_{\mathcal{A}}))$, and that \hat{H} preserves area. Thus we have,

$$(13) \quad Area(\hat{H}^n(\mathcal{A}) \cap \mathcal{B}) = \frac{|\hat{H}_*^n(\llbracket \gamma_{\mathcal{A}} \rrbracket) \cap \llbracket \gamma_{\mathcal{B}} \rrbracket|}{|H^n(hol(\gamma_{\mathcal{A}}) \wedge hol(\gamma_{\mathcal{B}}))|} Area(\mathcal{A}) Area(\mathcal{B}).$$

Here \hat{H}_* denotes the action of \hat{H} on the first homology group, $\llbracket \gamma_{\mathcal{A}} \rrbracket$ and $\llbracket \gamma_{\mathcal{B}} \rrbracket$ are the homology classes of the curves, and $H \in SL(2, \mathbb{R})$ is the derivative of \hat{H} . The denominator of this expression is easily understood, but the numerator will require some real work.

5.2. Homological Spaces. Our surfaces have infinite genus. Therefore, we must be careful by what we mean by homology.

We use $S_{\geq 1}$ to denote any of the homeomorphic surfaces S_c , when we only care about topology. We will be considering the homological space $H_1(S_{\geq 1} \setminus \Sigma, \mathbb{Z})$, which we define to be the linear space generated by finite weighted sums homology classes of closed curves in $S_{\geq 1}$ with the singularities removed.

Let $\{\mathcal{C}_{1,0}^i \mid i \in \mathbb{N}\}$ be the collection of horizontal cylinders ordered by increasing size and oriented rightward. Similarly, $\{\mathcal{C}_{1,1}^i \mid i \in \mathbb{N}\}$ will be the collection of cylinders whose core curves have slope 1 and oriented in the direction $(1, 1)$. We will use $\llbracket \mathcal{C}_{p,q}^i \rrbracket$ to denote the homology class corresponding to the core curve of cylinder $\mathcal{C}_{p,q}^i$ oriented in direction $(p, q) \in \mathbb{R}^2$.

Proposition 21 (Generators for homology). *Elements of $H_1(S_{\geq 1} \setminus \Sigma, \mathbb{Z})$ can be written uniquely as a finite linear combination of elements of $\{\llbracket \mathcal{C}_{1,0}^i \rrbracket\}_{i \in \mathbb{N}}$ and $\{\llbracket \mathcal{C}_{1,1}^i \rrbracket\}_{i \in \mathbb{N}}$.*

That is, cylinder which are of slope zero or one form a countable basis for $H_1(S_{\geq 1} \setminus \Sigma, \mathbb{Z})$. Figure 4 illustrates the decompositions into horizontal and slope one cylinders.

Proof. As in figure 1, we can cut $S_{\geq 1}$ into two polygonal pieces along countably many saddle connections. The resulting pieces are simply connected, so a homology class is determined by the signed number of times it crosses each of these saddle connections. Number the saddle connections $\{\sigma_i\}_{i \in \mathbb{Z}}$ counterclockwise from the point of view of the top piece so that the saddle connection of slope one in figure 1 is numbered

σ_0 . It is sufficient to show that there is a unique way to write the curve γ_i for $i \neq 0$ which enters the top piece through σ_0 then exits the top piece through σ_i then travels through the bottom piece to σ_0 and closes up. The only way to write $[[\gamma_i]]$ is by

$$[[\gamma_i]] = \begin{cases} \sum_{j=1}^{-i-1} [[\mathcal{C}_{1,1}^j]] - \sum_{j=1}^{-i} [[\mathcal{C}_{1,0}^j]] & \text{if } i < 0 \\ \sum_{j=1}^i [[\mathcal{C}_{1,1}^j]] - \sum_{j=1}^{-i} [[\mathcal{C}_{1,0}^j]] & \text{if } i > 0. \end{cases}$$

To see that this is the unique way to write $[[\gamma_i]]$, note that there is only one horizontal or slope one cylinder that intersects σ_0 , and only two of these cylinders intersect each of the remaining σ_i . \square

We will use the pair $\langle\langle \alpha, \beta \rangle\rangle$ where $\alpha = \langle \alpha_n \rangle_{n \in \mathbb{N}}$ and $\beta = \langle \beta_n \rangle_{n \in \mathbb{N}}$ are integral sequences which have bounded support (eventually all terms are zero) to denote the homology class

$$\langle\langle \alpha, \beta \rangle\rangle = \sum_{n \in \mathbb{N}} \alpha_n [[\mathcal{C}_{1,0}^n]] + \sum_{n \in \mathbb{N}} \beta_n [[\mathcal{C}_{1,1}^n]].$$

By the theorem, every element of $H_1(S_{\geq 1} \setminus \Sigma, \mathbb{Z})$ can be written uniquely in this way.

Given two homology classes, we can compute the algebraic intersection number between the two. This gives rise to an integral symplectic bilinear form on $H_1(S_{\geq 1} \setminus \Sigma, \mathbb{Z})$. It is not hard to check that $\cap : H_1(S_{\geq 1} \setminus \Sigma, \mathbb{Z}) \times H_1(S_{\geq 1} \setminus \Sigma, \mathbb{Z}) \rightarrow \mathbb{Z}$ is given by the formula

$$(14) \quad \langle\langle \alpha^x, \beta^x \rangle\rangle \cap \langle\langle \alpha^y, \beta^y \rangle\rangle = \sum_{n \in \mathbb{Z}} (\alpha_n^x \beta_n^y + \alpha_{n+1}^x \beta_n^y - \beta_n^x \alpha_n^y - \beta_n^x \alpha_{n+1}^y)$$

The translation surface structures on $S_{\geq 1}$, given by S_c for $c \geq 1$, yield more structure. The holonomy depends only on the homology class of the curve and acts linearly. Thus, for each $c \geq 1$ we have the holonomy maps $hol_c : H_1(S_{\geq 1} \setminus \Sigma, \mathbb{Z}) \rightarrow \mathbb{R}^2$. These maps are naturally cohomology classes, but we will prefer to think of them as homology classes by using the intersection form. For instance, the vertical holonomy of a curve can be computed applying the intersection pairing to the homology class of the curve and a (countable) sum of homology classes of the horizontal cylinders weighted by the cylinder's widths. This requires more than a finite weighted sum of generators, so we introduce the larger space $\bar{H}_1(S_{\geq 1} \setminus \Sigma, \mathbb{R}) \supset H_1(S_{\geq 1} \setminus \Sigma, \mathbb{R})$. We define $\bar{H}_1(S_{\geq 1} \setminus \Sigma, \mathbb{R})$ to be the space of all pairs of sequences $\langle\langle \alpha, \beta \rangle\rangle$ of the form $\alpha = \langle \alpha_n \in \mathbb{R} \rangle_{n \in \mathbb{N}}$ and $\beta = \langle \beta_n \in \mathbb{R} \rangle_{n \in \mathbb{N}}$. We have no condition of bounded support on elements of $\bar{H}_1(S_{\geq 1} \setminus \Sigma, \mathbb{R})$. Equation 14 extends \cap to a bilinear map

$$\cap : \bar{H}_1(S_{\geq 1} \setminus \Sigma, \mathbb{R}) \times H_1(S_{\geq 1} \setminus \Sigma, \mathbb{R}) \rightarrow \mathbb{R}.$$

Proposition 22. For all $c \geq 1$, define the linear map $\mathcal{L}_c : \mathbb{R}^2 \rightarrow \bar{H}_1(S_{\geq 1} \setminus \Sigma, \mathbb{R})$ by $\mathcal{L}_c : (a, b) \mapsto \langle\langle \alpha, \beta \rangle\rangle$, where $\langle\langle \alpha, \beta \rangle\rangle$ is given coordinatewise by polynomials in $c = \cos \theta$ as follows.

$$\alpha_n = (a - b) \text{Width}(\mathcal{C}_{1,0}^n) = (a - b) \sum_{j=-(n-1)}^{n-1} \cos j\theta$$

$$\beta_n = b \text{Width}(\mathcal{C}_{1,1}^n) \sqrt{2} = \frac{(2b) \sum_{j=-(n-1)}^n \cos j\theta}{1 + \cos \theta}$$

\mathcal{L}_c satisfies the identity $\mathcal{L}_c(a, b) \cap \llbracket x \rrbracket = (a, b) \wedge \text{hol}_c(\llbracket x \rrbracket)$ for all $(a, b) \in \mathbb{R}^2$.

Remark 23. In the above proposition, the quantities α_n and β_n should be viewed as polynomials in $c = \cos \theta$ (e.g. $\cos(-2\theta) = 2 \cos^2 \theta - 1 = 2c^2 - 1$). The term $1 + \cos \theta$ in the denominator of β_n cancels with a factor from the numerator. Since all α_n and β_n are polynomials in c , it makes sense to define the expression $\mathcal{L}_c(a, b)$ for all $c \in \mathbb{C}$.

Proof of proposition 22. It is sufficient to prove the statement on pairings basis of \mathbb{R}^2 . We choose the basis $\{(1, 0), (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})\}$. The quantity $(1, 0) \wedge \text{hol}_c(\llbracket x \rrbracket)$ measures the y -coordinate of $\text{hol}_c(\llbracket x \rrbracket)$. As mentioned in the paragraph before the proposition, this quantity may be computed by intersecting with the homology class

$$\sum_{n=1}^{\infty} \text{Width}(\mathcal{C}_{1,0}^n) \llbracket \mathcal{C}_{1,0}^n \rrbracket.$$

After some trigonometric simplifications, it can be checked that this is the same as $\mathcal{L}_c(1, 0)$. Similarly, $(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}) \wedge \text{hol}_c(\llbracket x \rrbracket)$ measures distance in the slope -1 direction. This can be computed by intersecting with

$$\sum_{n=1}^{\infty} \text{Width}(\mathcal{C}_{1,1}^n) \llbracket \mathcal{C}_{1,1}^n \rrbracket.$$

Again, this is the same as $\mathcal{L}_c(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$. \square

We can define the first cohomological group $H^1(S_{\geq 1} \setminus \Sigma, \mathbb{R})$ as the set of all linear maps $\phi : H_1(S_{\geq 1} \setminus \Sigma, \mathbb{R}) \rightarrow \mathbb{R}$. Via \cap , there is a natural map from $\Phi : \bar{H}_1(S_{\geq 1} \setminus \Sigma, \mathbb{R}) \rightarrow H^1(S_{\geq 1} \setminus \Sigma, \mathbb{R})$ namely given $\llbracket x \rrbracket \in \bar{H}_1(S_{\geq 1} \setminus \Sigma, \mathbb{R})$ take the map

$$\Phi(\llbracket x \rrbracket) : H_1(S_{\geq 1} \setminus \Sigma, \mathbb{R}) \rightarrow \mathbb{R} : \llbracket y \rrbracket \mapsto \llbracket x \rrbracket \cap \llbracket y \rrbracket.$$

But Φ is not injective. It can be checked that Φ has a one dimensional kernel spanned by the element

$$(15) \quad \llbracket z \rrbracket = \langle\langle \langle 1, -1, 1, -1, \dots \rangle, \langle 0, 0, 0, \dots \rangle \rangle\rangle.$$

5.3. Action of automorphisms on homology. By theorem 2, the actions of the affine automorphism groups $Aut(S_c)$ on $S_{\geq 1}$ are the same up to isotopy. Therefore, it makes sense to use $Aut(S_{\geq 1})$ to denote this group action, when we act on homology. We will describe the action of generators of $Aut(S_{\geq 1})$ on $H_1(S_{\geq 1} \setminus \Sigma, \mathbb{Z})$.

If $\{\gamma_i\}_{i \in \Lambda}$ is a countable collection of disjoint curves on a surface S , then the action of a right Dehn twist on homology is given by the map

$$H_1(S \setminus \Sigma, \mathbb{Z}) \rightarrow H_1(S \setminus \Sigma, \mathbb{Z}) : [x] \mapsto [x] + \sum_{i \in \Lambda} ([\gamma_i] \cap [x]) [\gamma_i],$$

which is independent of the orientations assigned to the γ_i .

The action of the element $\hat{D} \in Aut(S_{\geq 1})$ on homology is given by

$$\begin{aligned} \hat{D}_* & : H_1(S_{\geq 1} \setminus \Sigma, \mathbb{Z}) \rightarrow H_1(S_{\geq 1} \setminus \Sigma, \mathbb{Z}) \\ & : [x] \mapsto [x] + \sum_{i \in \mathbb{N}} ([\mathcal{C}_{1,0}^i] \cap [x]) [\mathcal{C}_{1,0}^i]. \end{aligned}$$

If we take $\langle\langle \alpha, \beta \rangle\rangle \in H_1(S_{\geq 1} \setminus \Sigma, \mathbb{Z})$ and set $\langle\langle \alpha', \beta' \rangle\rangle = \hat{D}_*(\langle\langle \alpha, \beta \rangle\rangle)$ then

$$(16) \quad \alpha'_n = \alpha_n + \beta_{n-1} + \beta_n \quad \text{and} \quad \beta'_n = \beta_n,$$

where we interpret $\beta_0 = 0$ (since $0 \notin \mathbb{N}$).

Similarly, the action of $\hat{E} \in Aut(S_{\geq 1})$ on homology is

$$\begin{aligned} \hat{E}_* & : H_1(S_{\geq 1} \setminus \Sigma, \mathbb{Z}) \rightarrow H_1(S_{\geq 1} \setminus \Sigma, \mathbb{Z}) \\ & : [x] \mapsto [x] + \sum_{i \in \mathbb{N}} ([\mathcal{C}_{1,1}^i] \cap [x]) [\mathcal{C}_{1,1}^i]. \end{aligned}$$

If we set $\langle\langle \alpha', \beta' \rangle\rangle = \hat{E}_*(\langle\langle \alpha, \beta \rangle\rangle)$, then

$$(17) \quad \alpha'_n = \alpha_n \quad \text{and} \quad \beta'_n = \beta_n - \alpha_n - \alpha_{n+1}.$$

Finally, we describe the action of $\hat{A} \in Aut(S_{\geq 1})$ on homology. If $\langle\langle \alpha', \beta' \rangle\rangle = \hat{A}_*(\langle\langle \alpha, \beta \rangle\rangle)$ then

$$(18) \quad \alpha'_n = -\alpha_n - \beta_{n-1} - \beta_n \quad \text{and} \quad \beta'_n = \beta_n.$$

The action of every $\hat{G} \in Aut(S_{\geq 1})$ on homology is *sparse* in the sense that the formula for the α'_n and β'_n of terms $\langle\langle \alpha', \beta' \rangle\rangle = \hat{G}_*(\langle\langle \alpha, \beta \rangle\rangle)$ only involve terms of the form α_m and β_m for m within some constant $k(\hat{G})$ of n ($|m - n| < k$). Thus, the action extends to $\bar{H}_1(S \setminus \Sigma, \mathbb{Z})$. Furthermore, it must be that $\hat{G}_*([y]) \cap \hat{G}_*([x]) = [y] \cap [x]$ for all $[x] \in H_1(S \setminus \Sigma, \mathbb{Z})$ and $y \in \bar{H}_1(S \setminus \Sigma, \mathbb{Z})$.

5.4. Invariant Planes. Inside the action of $Aut(S_{\geq 1})$ on the homological group $\bar{H}_1(S_{\geq 1} \setminus \Sigma, \mathbb{R})$, there are embedded copies of the group actions on \mathbb{R}^2 given by the Veech groups $\Gamma(S_c) = \mathcal{G}_c^\pm$. Actually, the groups \mathcal{G}_c make sense for all $c \in \mathbb{R}$ as representations of the abstract group \mathcal{G}^\pm into $SL^\pm(2, \mathbb{R})$. See theorem 1 for the definition of \mathcal{G}_c . Given

a $G \in \mathcal{G}^\pm$, we have the corresponding $\hat{G} \in \text{Aut}(S_{\geq 1})$ as well as the Veech group elements $G_c \in \mathcal{G}_c$ for all $c \in \mathbb{R}$.

Theorem 24. *For all $c \in \mathbb{R}$, the planes spanned by h_c and v_c are invariant. Furthermore, the linear map $\mathcal{L}_c : \mathbb{R}^2 \rightarrow \bar{H}_1(S_{\geq 1} \setminus \Sigma, \mathbb{R})$ defined in proposition 22 completes the commutative diagram*

$$\begin{array}{ccc} \mathbb{R}^2 & \xrightarrow{G_c} & \mathbb{R}^2 \\ \downarrow \mathcal{L}_c & & \downarrow \mathcal{L}_c \\ \bar{H}_1(S_{\geq 1} \setminus \Sigma, \mathbb{R}) & \xrightarrow{\hat{G}_*} & \bar{H}_1(S_{\geq 1} \setminus \Sigma, \mathbb{R}) \end{array}$$

for all $G \in \mathcal{G}^\pm$.

Proof. By proposition 22, it follows that for all $\llbracket x \rrbracket \in H_1(S_{\geq 1} \setminus \Sigma, \mathbb{R})$, $(a, b) \in \mathbb{R}^2$, and $G \in \mathcal{G}^\pm$ we have

$$\mathcal{L}_c(a, b) \cap \hat{G}_*^{-1}(\llbracket x \rrbracket) = (a, b) \wedge \text{hol}_c(\hat{G}_*^{-1}(\llbracket x \rrbracket)) = (a, b) \wedge G_c^{-1}(\text{hol}_c(\llbracket x \rrbracket))$$

By applying the invariance of \wedge under G_c and the invariance of \cap under \hat{G}_* , we deduce

$$G_c(a, b) \wedge \text{hol}_c(\llbracket x \rrbracket) = \hat{G}_*(\mathcal{L}_c(a, b)) \cap \llbracket x \rrbracket.$$

Thus, by an application of proposition 22, to the left side we have

$$\mathcal{L}_c(G_c(a, b)) \cap \llbracket x \rrbracket = \hat{G}_*(\mathcal{L}_c(a, b)) \cap \llbracket x \rrbracket.$$

This equation says that the actions of $\hat{G}_*(\mathcal{L}_c(a, b)) \cap$ and $\mathcal{L}_c(G_c(a, b)) \cap$ on homology are the same. Unfortunately, by the remarks before the definition of $\llbracket z \rrbracket \in \bar{H}_1(S_{\geq 1} \setminus \Sigma, \mathbb{Z})$ in equation 15, this only implies that $\hat{G}_*(\mathcal{L}_c(a, b)) = \mathcal{L}_c(G_c(a, b)) + s \llbracket z \rrbracket$ for some $s \in \mathbb{R}$. It must therefore be explicitly checked that $s = 0$ for the generators $G \in \{A, D, E\}$.

For the readers sake, we only check the case of $G = A$. Let $\langle\langle \alpha, \beta \rangle\rangle = \hat{A}_*(\mathcal{L}_c(a, b))$ and $\langle\langle \alpha', \beta' \rangle\rangle = \mathcal{L}_c(A_c(a, b))$. It is sufficient to check that $\alpha_1 = \alpha'_1$. This is a straightforward calculation utilizing the \hat{A}_* action given in equation 18, the definition of A_c given in theorem 1, and the definition of \mathcal{L}_c given in proposition 22. We compute $\alpha_1 = -a - b = \alpha'_1$. \square

We will now show that we can write every element of $\llbracket x \rrbracket \in H_1(S_{\geq 1} \setminus \Sigma, \mathbb{R})$ as an integral of vectors in the invariant planes $\mathcal{L}_{\cos \theta}$ for $\theta \in [-\pi, \pi]$. That is, we will find functions $r, s : [-\pi, \pi] \rightarrow \mathbb{R}$ so that

$$\llbracket x \rrbracket = \int_{-\pi}^{\pi} \mathcal{L}_{\cos \theta}(r(\theta), s(\theta)) d\theta.$$

We interpret such an integral *coordinatewise*. That is, we integrate each coordinate independently. Then theorem 24 implies that for all $\hat{G} \in \text{Aut}(S_{\geq 1})$,

$$(19) \quad \hat{G}_*([\![x]\!]) = \int_{-\pi}^{\pi} \mathcal{L}_{\cos \theta} \left(G_{\cos \theta}(r(\theta), s(\theta)) \right) d\theta.$$

Indeed, it turns out that there is a natural choice for r and s as polynomials in $c = \cos \theta$.

Let \mathbf{P} denote the space of all polynomials. \mathbf{P}^2 will denote the space of all pairs of polynomials.

Theorem 25. *Consider the linear embedding $\psi : H_1(S_{\geq 1} \setminus \Sigma, \mathbb{R}) \rightarrow \mathbf{P}^2$ determined by the following images of the basis elements, which are polynomials in $c = \cos \theta$.*

$$\psi : [\![\mathcal{C}_{1,0}^n]\!] \mapsto (2 \cos((n-1)\theta) - 2 \cos(n\theta), 0)$$

$$\psi : [\![\mathcal{C}_{1,1}^n]\!] \mapsto ((\cos((n-1)\theta) - \cos((n+1)\theta))(1, 1)$$

This embedding satisfies the following for all $[\![x]\!] \in H_1(S_{\geq 1} \setminus \Sigma, \mathbb{R})$.

- (1) $[\![x]\!] = \frac{1}{4\pi} \int_{-\pi}^{\pi} \mathcal{L}_{\cos \theta} (\psi([\![x]\!]) (\cos \theta)) d\theta$ coordinatewise.
- (2) $\psi([\![x]\!])(1) = (0, 0)$ and $\frac{d}{d\theta} \psi([\![x]\!]) (\cos \theta)|_{\theta=0} = (0, 0)$.
- (3) $hol_1([\![x]\!]) = \frac{d^2}{d\theta^2} \psi([\![x]\!]) (\cos \theta)|_{\theta=0}$.

Proof. We first prove statement 1. It is enough to prove this for our basis of $H_1(S_{\geq 1} \setminus \Sigma, \mathbb{R})$. We take the first case of $[\![\mathcal{C}_{1,0}^n]\!]$. Let

$$\langle\langle \alpha, \beta \rangle\rangle = \frac{1}{4\pi} \int_{-\pi}^{\pi} \mathcal{L}_{\cos \theta} (2 \cos((n-1)\theta) - 2 \cos(n\theta)) d\theta$$

We wish to show $\langle\langle \alpha, \beta \rangle\rangle = [\![\mathcal{C}_{1,0}^n]\!]$. Then

$$\alpha_k = \frac{1}{4\pi} \int_{-\pi}^{\pi} (2 \cos((n-1)\theta) - 2 \cos(n\theta)) \sum_{j=-(n-1)}^{n-1} \cos j\theta d\theta.$$

Using ideas from Fourier series, it is not hard to see that $\alpha_k = 1$ if $n = k$ and $\alpha_k = 0$ otherwise. The β_k must be zero, because $\mathcal{L}_{\cos \theta}(1, 0)$ has zero β portion, regardless of θ . Now we take the second case of $[\![\mathcal{C}_{1,1}^n]\!]$. This time let

$$\langle\langle \alpha, \beta \rangle\rangle = \frac{1}{4\pi} \int_{-\pi}^{\pi} (\cos((n-1)\theta) - \cos((n+1)\theta)) \mathcal{L}_{\cos \theta}(1, 1) d\theta.$$

The α coordinates of $\mathcal{L}_{\cos \theta}(1, 1)$ are all zero, so for all k we have $\alpha_k = 0$. We also have

$$\beta_k = \frac{1}{4\pi} \int_{-\pi}^{\pi} (\cos((n-1)\theta) - \cos((n+1)\theta)) \left(\frac{2 \sum_{j=-(n-1)}^n \cos j\theta}{1 + \cos \theta} \right) d\theta.$$

Trigonometric tricks can be used to show that for all $n \geq 1$,

$$\frac{\cos((n-1)\theta) - \cos((n+1)\theta)}{1 + \cos \theta} = 2 \sum_{j=-(n-1)}^n (-1)^{j+n+1} \cos j\theta$$

Interestingly, the expression inside the integral reduces to a symmetric expression in n and k .

$$\beta_k = \frac{1}{4\pi} \int_{-\pi}^{\pi} 2 \cos((n-k)\theta) - 2 \cos((n+k)\theta) d\theta.$$

So $\beta_k = 1$ when $n = k$ and $\beta_k = 0$ otherwise.

Statement 2 is a trivial calculation, so we omit it. Statement 3 is trivial as well.

$$\begin{aligned} \text{hol}_1(\llbracket \mathcal{C}_{1,0}^n \rrbracket) &= (4n - 2, 0) = \frac{d^2}{d\theta^2} (2 \cos((n-1)\theta) - 2 \cos(n\theta), 0)|_{\theta=0} \\ \text{hol}_1(\llbracket \mathcal{C}_{1,1}^n \rrbracket) &= (4n, 4n) = \frac{d^2}{d\theta^2} ((\cos((n-1)\theta) - \cos((n+1)\theta))(1, 1)|_{\theta=0} \end{aligned}$$

□

5.5. An asymptotic formula. We say two sequences, $\langle r_n \rangle$ and $\langle s_n \rangle$, are *asymptotic* if $\lim_{n \rightarrow \infty} \frac{r_n}{s_n} = 1$. We denote this by $r_n \asymp s_n$.

Lemma 26 (Multiplicative asymptotic growth). *Let $\epsilon > 0$, $a > 0$, and $f, g : [-\epsilon, \epsilon] \rightarrow \mathbb{R}$ be C^2 functions satisfying*

- (1) $f(0) = \frac{d}{dx} f(0) = 0$ and $\frac{d^2}{dx^2} f(0) = 2$.
- (2) $g(0) = 1$, $\frac{d}{dx} g(0) = 0$, and $\frac{d^2}{dx^2} g(0) = -2a$.
- (3) $|g(x)| < 1$ on $[-\epsilon, 0) \cup (0, \epsilon]$.

Then, the sequences given by

$$r_n = \int_{-\epsilon}^{\epsilon} f(x) g^n(x) dx \quad \text{and} \quad s_n = \frac{\sqrt{\pi}}{2a^{\frac{3}{2}} n^{\frac{3}{2}}}$$

are asymptotic.

We prove this multiplicative asymptotic growth lemma in section 6.1 of the appendix.

5.6. Asymptotic growth of homology classes. We will say a sequence of homology classes $\llbracket x_m \rrbracket = \langle\langle \alpha^m, \beta^m \rangle\rangle$ in $\bar{H}_1(S_{\geq 1} \setminus \Sigma, \mathbb{R})$ converges *coordinatewise* to $\llbracket x_\infty \rrbracket = \langle\langle \alpha^\infty, \beta^\infty \rangle\rangle \in \bar{H}_1(S_{\geq 1} \setminus \Sigma, \mathbb{R})$ if for every n

$$\lim_{m \rightarrow \infty} \alpha_n^m = \alpha_n^\infty \quad \text{and} \quad \lim_{m \rightarrow \infty} \beta_n^m = \beta_n^\infty$$

An element $\hat{G} \in \text{Aut}(S_{\geq 1})$ acts *hyperbolically* on S_1 if there is an eigenvalue $\lambda_1 > 1$ of $G_1 = \mathbf{D}(\hat{G}) \in \Gamma(S_1)$. Let $\mathbf{v}_1^+ \in \mathbb{R}^2$ be a unit

length eigenvector with the eigenvalue λ_1 , and let $\mathbf{v}_1^- \in \mathbb{R}^2$ be a unit eigenvector with eigenvalue $\pm 1/\lambda_1$.

Theorem 27 (Asymptotics of Homology). *There is a constant $\kappa_G > 0$ depending only on \hat{G} so that given any homology class $\llbracket x \rrbracket \in H_1(S_{\geq 1} \setminus \Sigma, \mathbb{R})$, the sequence of homology classes*

$$\llbracket x_m \rrbracket = \frac{m^{\frac{3}{2}} \hat{G}_*^m(\llbracket x \rrbracket)}{\lambda_1^m} \in H_1(S_{\geq 1} \setminus \Sigma, \mathbb{R})$$

converge coordinatewise as $m \rightarrow \infty$ to

$$\llbracket x_\infty \rrbracket = \kappa_G \mathcal{L}_1(\text{proj}_{\mathbf{v}_1^-, \mathbf{v}_1^+} \text{hol}_1(\llbracket x \rrbracket)),$$

where $\text{proj}_{\mathbf{v}_1^-, \mathbf{v}_1^+} \text{hol}_1(\llbracket x \rrbracket)$ denotes the projection of the holonomy vector in the direction \mathbf{v}_1^- onto the direction \mathbf{v}_1^+ . Furthermore,

$$\kappa_G = \frac{1}{2\sqrt{\pi}} \left(\frac{d}{dc} [2 \log \lambda_c] \Big|_{c=1} \right)^{-\frac{3}{2}}$$

where \log is the natural logarithm and λ_c denotes the greatest eigenvalue of G_c . Note, $2 \log \lambda_c$ is geometrically the translation distance of the action of G_c on the hyperbolic plane.

The first idea is to rewrite $\llbracket x \rrbracket = \langle\langle \alpha, \beta \rangle\rangle$ using theorem 25. There is a polynomial $\psi(\llbracket x \rrbracket) : \mathbb{R} \rightarrow \mathbb{R}^2$ in the variable $c = \cos \theta$ such that

$$\llbracket x \rrbracket = \frac{1}{4\pi} \int_{-\pi}^{\pi} \mathcal{L}_{\cos \theta}(\psi(\llbracket x \rrbracket)(\cos \theta)) d\theta.$$

Then theorem 24 and equation 19 tell us that we can write

$$(20) \quad \llbracket x_m \rrbracket = \frac{m^{\frac{3}{2}}}{4\pi \lambda_1^m} \int_{-\pi}^{\pi} \mathcal{L}_{\cos \theta} \circ G_{\cos \theta}^m(\psi(\llbracket x \rrbracket)(\cos \theta)) d\theta.$$

Now we will rewrite the action of $G_{\cos \theta}$ in a different basis depending on θ . Let λ_c denote the largest eigenvalue of $G_{\cos \theta}$ and let \mathbf{v}_c^+ be a corresponding unit eigenvector, and \mathbf{v}_c^- be a unit eigenvector with eigenvalue $\pm \lambda_c^{-1}$. These choices only make sense so long as $|\text{Trace}(G_{\cos \theta})| > 2$. The sign \pm is invariant under θ , since it determines whether \hat{G} is orientation preserving or reversing. Since $G_{\cos \theta}$ varies polynomially in $c = \cos \theta$, by restricting to a small enough interval, $\delta \leq c \leq 1$, we may assume all these quantities are C^∞ in $c = \cos \theta$. Then we can find C^∞ functions $\mathbf{p}, \mathbf{q} : \cos^{-1}([-\epsilon, \epsilon]) \rightarrow \mathbb{R}^2$

$$(21) \quad \psi(\llbracket x \rrbracket)(\cos \theta) = \mathbf{p}(\cos \theta) + \mathbf{q}(\cos \theta)$$

with $\mathbf{p}(\cos \theta)$ parallel to the expanding direction \mathbf{v}_c^+ and $\mathbf{q}(\cos \theta)$ parallel to \mathbf{v}_c^- . Formulas for these functions exist in terms of the polynomials making up $G_{\cos \theta}$ and $\psi(\llbracket x \rrbracket)(\cos \theta)$. $\mathbf{p}(\cos \theta)$ is the projection of

$\psi(\llbracket x \rrbracket)(\cos \theta)$ in the \mathbf{v}_c^- direction onto the \mathbf{v}_c^+ direction, and $\mathbf{q}(\cos \theta)$ is the reverse. By choosing ϵ sufficiently small, \mathbf{p} and \mathbf{q} may be made C^∞ functions of $\cos \theta$ on their domains. This allows us to write the simplification

$$G_{\cos \theta}^m(\psi(\llbracket x \rrbracket)(\cos \theta)) = \lambda_c^m \mathbf{p}(\cos \theta) + (\pm \lambda_c)^{-m} \mathbf{q}(\cos \theta).$$

In summary, we may break up equation 20 as a sum of three integrals

$$(22) \quad \llbracket x_m \rrbracket = \frac{m^{\frac{3}{2}}}{4\pi\lambda_1^m} \left(\int_{-\epsilon}^{\epsilon} \lambda_c^m \mathcal{L}_{\cos \theta}(\mathbf{p}(\cos \theta)) d\theta + \int_{-\epsilon}^{\epsilon} (\pm \lambda_c)^{-m} \mathcal{L}_{\cos \theta}(\mathbf{q}(\cos \theta)) d\theta + \int_{[-\pi, -\epsilon] \cup [\epsilon, \pi]} \mathcal{L}_{\cos \theta} \circ G_{\cos \theta}^m(\psi(\llbracket x \rrbracket)(\cos \theta)) d\theta \right)$$

We will show that in the limit the first integral contributes, but the other two do not. The first integral contributes by the multiplicative asymptotic growth lemma. The lemmas below are the keys to these facts.

Lemma 28 (Derivative of the eigenvalue). *Let $\hat{G} \in \text{Aut}(S_{\geq 1})$ be chosen so that \hat{G} acts hyperbolically on S_1 . Let λ_c be the largest eigenvalue of G_c . Then*

$$\left. \frac{d}{dc} \lambda_c \right|_{c=1} > 0.$$

Lemma 29 (Eigenvalue upper bound). *Let $\hat{G} \in \text{Aut}(S_{\geq 1})$ be chosen so that \hat{G} acts hyperbolically on S_1 . Let λ_c be the largest eigenvalue of G_c . Then $|\lambda_c| < \lambda_1$ for all $-1 \leq c < 1$.*

We will prove these lemmas in section 6.2.

Proof of theorem 27. We continue the discussion below the statement of the theorem, where we left off with equation 22.

We will compute the contribution of the first integral of equation 22,

$$I_1(m) = \frac{m^{\frac{3}{2}}}{4\pi} \int_{-\epsilon}^{\epsilon} \left(\frac{\lambda_c}{\lambda_1} \right)^m \mathcal{L}_{\cos \theta}(\mathbf{p}(\cos \theta)) d\theta.$$

We will show that we can apply the asymptotic growth lemma. Set

$$f(\theta) = \mathcal{L}_{\cos \theta}(\mathbf{p}(\cos \theta)).$$

In order to apply the lemma, we must compute some derivatives. First we will examine $\mathbf{p}(\cos \theta)$. We may write

$$\mathbf{p}(\cos \theta) = (\psi(\llbracket x \rrbracket)(\cos \theta) \cdot \mathbf{v}_c^+) \mathbf{v}_c^+$$

Clearly $\mathbf{p}(1) = 0$ since $\psi(\llbracket x \rrbracket)(1) = 0$, by the first part of statement 2 of theorem 25. Also $\left. \frac{d}{d\theta} \mathbf{p}(\cos \theta) \right|_{\theta=0} = (0, 0)$ because \mathbf{p} is a function of

$\cos \theta$. We will also need the second derivative. Using the product rule for derivatives and applying theorem 25, we can compute

$$\frac{d^2}{d\theta^2} \mathbf{p}(\cos \theta) \Big|_{\theta=0} = \left(\frac{d^2}{d\theta^2} \psi(\llbracket x \rrbracket) \right) (\cos \theta) \cdot \mathbf{v}_1^+ \mathbf{v}_1^+ = (\text{hol}_1(\llbracket x \rrbracket)) \cdot \mathbf{v}_1^+ \mathbf{v}_1^+.$$

(The product rule gives several terms, but this is the only one that contributes, since the zeroth and first derivative of $\psi(\llbracket x \rrbracket)$ is zero.) Note, this is precisely the quantity $\text{proj}_{\mathbf{v}_1^-, \mathbf{v}_1^+} \text{hol}_1(\llbracket x \rrbracket)$ from the statement of the theorem. We return to analyzing $f(\theta)$. By proposition 22, each coordinate i of $f(\theta)$ may be written in the form

$$f_i(\theta) = \rho_i(\cos \theta) L_i(\mathbf{p}(\cos \theta)),$$

where $\rho_i(\cos \theta)$ is a polynomial in $\cos \theta$ and $L_i : \mathbb{R}^2 \rightarrow \mathbb{R}$ is a linear map independent of $\cos \theta$. Since the zeroth and first derivatives of \mathbf{p} vanish, so do the zeroth and first of f_i . Thus, the zeroth and first derivatives of $f(\theta)$ vanish. Moreover, we can see

$$\frac{d^2}{d\theta^2} f_i(\theta) \Big|_{\theta=0} = \rho_i(\cos \theta) L_i \left(\frac{d^2}{d\theta^2} \mathbf{p}(\cos \theta) \right) = \rho_i(\cos \theta) L_i(\text{proj}_{\mathbf{v}_1^-, \mathbf{v}_1^+} \text{hol}_1(\llbracket x \rrbracket)).$$

This implies that the following equation is true coordinatewise.

$$\frac{d^2}{d\theta^2} f(\theta) = \mathcal{L}_{\cos \theta} \left(\frac{d^2}{d\theta^2} \mathbf{p}(\cos \theta) \right) = \mathcal{L}_{\cos \theta}(\text{proj}_{\mathbf{v}_1^-, \mathbf{v}_1^+} \text{hol}_1(\llbracket x \rrbracket)).$$

Thus, the function $f(\theta)$ satisfies the first condition of the asymptotic growth lemma coordinatewise up to a scalar multiple.

Now set $g(\theta) = \frac{\lambda_c}{\lambda_1}$. Clearly, $g(0) = 1$. Also $\frac{d}{d\theta} g(0) = 0$, because g is a function of $\cos \theta$. For the same reason, we have

$$\frac{d^2}{d\theta^2} g(0) = -\frac{d}{dc} \left(\frac{\lambda_c}{\lambda_1} \right) \Big|_{c=1} = \frac{-\frac{d}{dc} \lambda_c \Big|_{c=1}}{\lambda_1} = -\frac{d}{dc} \log \lambda_c \Big|_{c=1} < 0,$$

with the last inequality following from lemma 28. Thus $g(\theta)$ satisfies the second condition of the asymptotic growth lemma. Further, lemma 29 implies $g(\theta)$ satisfies the third condition of the asymptotic growth lemma.

By the asymptotic growth lemma,

$$\begin{aligned} I_1(m) &= \frac{m^{\frac{3}{2}}}{4\pi} \int_{-\epsilon}^{\epsilon} g(\theta)^m f(\theta) d\theta = \frac{\frac{1}{2} \mathcal{L}_{\cos \theta}(\text{proj}_{\mathbf{v}_1^-, \mathbf{v}_1^+} \text{hol}_1(\llbracket x \rrbracket))}{8\sqrt{\pi} \left(\frac{1}{2} \frac{d}{dc} \log \lambda_c \Big|_{c=1} \right)^{\frac{3}{2}}} \\ &= \frac{\mathcal{L}_{\cos \theta}(\text{proj}_{\mathbf{v}_1^-, \mathbf{v}_1^+} \text{hol}_1(\llbracket x \rrbracket))}{2\sqrt{\pi} \left(2 \frac{d}{dc} \log \lambda_c \Big|_{c=1} \right)^{\frac{3}{2}}} = \kappa_H \mathcal{L}_{\cos \theta}(\text{proj}_{\mathbf{v}_1^-, \mathbf{v}_1^+} \text{hol}_1(\llbracket x \rrbracket)), \end{aligned}$$

which is the limit $\llbracket x_\infty \rrbracket$ claimed by the theorem.

The second integral from equation 22 may be written as

$$I_2(m) = \frac{1}{4\pi} \int_{-\epsilon}^{\epsilon} \left(\frac{1}{\pm \lambda_c \lambda_1} \right)^m m^{\frac{3}{2}} \mathcal{L}_{\cos \theta}(\mathbf{q}(\cos \theta)) d\theta.$$

The function inside the integral goes to zero pointwise in every coordinate, since $(\frac{1}{\lambda_c \lambda_1})^m$, is exponentially decaying, while $m^{\frac{3}{2}}$ grows polynomially. Thus, this term does not contribute to equation 22.

Finally, we consider the third integral from equation 22

$$I_3(m) = \frac{1}{4\pi} \int_{[-\pi, -\epsilon] \cup [\epsilon, \pi]} \mathcal{L}_{\cos \theta} \left(\frac{m^{\frac{3}{2}}}{\lambda_1^m} G_{\cos \theta}^m(\psi(\llbracket x \rrbracket)(\cos \theta)) \right) d\theta.$$

The function inside this integral also goes to zero pointwise in every coordinate. By lemma 29, for any θ in the domain of integration with $\text{Trace}(G_{\cos \theta})$, the exponential growth of the absolute values of entries in the sequence of matrices $G_{\cos \theta}^m$ must be slower than λ_1 . Moreover, by continuity of $G_{\cos \theta}$ and by compactness of the domain of integration, there must be a $\lambda' < \lambda_1$ so that the exponential growth of entries of $G_{\cos \theta}^m$ must be slower than by a factor of λ' . The entries of $\frac{m^{\frac{3}{2}}}{\lambda_1^m} G_{\cos \theta}^m$ decay exponentially. Now, if $G_{\cos \theta}$ is parabolic then entries of $G_{\cos \theta}^m$ grow at most linearly. Thus, $\frac{m^{\frac{3}{2}}}{\lambda_1^m} G_{\cos \theta}^m$ decays exponentially in this case too. Finally, $G_{\cos \theta}$ may be elliptic. Then the entries of $G_{\cos \theta}^m$ are bounded close to zero. In this case, $\frac{m^{\frac{3}{2}}}{\lambda_1^m} G_{\cos \theta}^m$ decays exponentially also. \square

5.7. Asymptotics of cylinder intersections. By applying theorem 27 to proposition 20, we obtain the following theorem.

Theorem 30 (Asymptotics of cylinder intersections). *Let \mathcal{A} and \mathcal{B} be any two cylinders on S_1 , then for any $\hat{G} \in \text{Aut}(S_{\geq 1})$ acting hyperbolically on S_1 ,*

$$\lim_{m \rightarrow \infty} m^{\frac{3}{2}} \text{Area}(\hat{G}^m(\mathcal{A}) \cap \mathcal{B}) = \kappa_G \text{Area}(\mathcal{A}) \text{Area}(\mathcal{B}),$$

with κ_G as in theorem 27.

Proof. By proposition 20, the limit is equal to

$$L = \lim_{m \rightarrow \infty} m^{\frac{3}{2}} \frac{|\hat{G}_*^m(\llbracket \gamma_{\mathcal{A}} \rrbracket) \cap \llbracket \gamma_{\mathcal{B}} \rrbracket|}{|G_1^m(\text{hol}(\gamma_{\mathcal{A}})) \wedge \text{hol}(\gamma_{\mathcal{B}})|} \text{Area}(\mathcal{A}) \text{Area}(\mathcal{B}).$$

Let $\text{proj}_{\mathbf{v}_-, \mathbf{v}_+} \text{hol}_1(\llbracket \gamma_{\mathcal{A}} \rrbracket)$ denote the projection of the holonomy vector in the direction of the compressing direction of G_1 onto the expanding direction of G_1 . By theorem 27, we may rewrite the numerator

using the asymptotic form of $\hat{G}_*^m(\llbracket\gamma_{\mathcal{A}}\rrbracket) \cap \llbracket\gamma_{\mathcal{B}}\rrbracket$. And the term in the denominator is asymptotic to $\lambda^m \text{proj}_{\mathbf{v}_-, \mathbf{v}_+} \text{hol}_1(\llbracket x \rrbracket) \wedge \text{hol}_1(\gamma_{\mathcal{B}})$. Thus,

$$L = \lim_{m \rightarrow \infty} \frac{|\lambda^m \kappa_G(\mathcal{L}_1(\text{proj}_{\mathbf{v}_-, \mathbf{v}_+} \text{hol}_1(\llbracket\gamma_{\mathcal{A}}\rrbracket)) \cap \llbracket\gamma_{\mathcal{B}}\rrbracket)|}{|\lambda^m \text{proj}_{\mathbf{v}_-, \mathbf{v}_+} \text{hol}_1(\llbracket\gamma_{\mathcal{A}}\rrbracket) \wedge \text{hol}_1(\gamma_{\mathcal{B}})|} \text{Area}(\mathcal{A}) \text{Area}(\mathcal{B}).$$

Finally, $\mathcal{L}_1(\text{proj}_{\mathbf{v}_-, \mathbf{v}_+} \text{hol}_1(\llbracket\gamma_{\mathcal{A}}\rrbracket)) \cap \llbracket\gamma_{\mathcal{B}}\rrbracket = \text{proj}_{\mathbf{v}_-, \mathbf{v}_+} \text{hol}_1(\llbracket\gamma_{\mathcal{A}}\rrbracket) \wedge \text{hol}_1(\gamma_{\mathcal{B}})$, by proposition 22. Furthermore, $\text{proj}_{\mathbf{v}_-, \mathbf{v}_+} \text{hol}_1(\llbracket\gamma_{\mathcal{A}}\rrbracket)$ points in an irrational direction while by corollary 14, $\text{hol}_1(\gamma_{\mathcal{B}})$ points in a rational direction. Hence, the wedge of the pair must be non-zero. So, we can cancel and get the limit we claimed. \square

We have the following corollary on the action of a hyperbolic element $\hat{G} \in \text{Aut}(S_{\geq 1})$ on S_1 .

Corollary 31. *The action of \hat{G} is not recurrent with respect to Lebesgue measure.*

Proof. If the \hat{G} action were recurrent with respect to Lebesgue measure, μ , then every measurable set A would satisfy

$$\sum_{m \in \mathbb{N}} \mu(\hat{G}^m(A) \cap A) = \infty.$$

But for $A = \mathcal{A}$ a cylinder, $\mu(\hat{G}^m(\mathcal{A}) \cap \mathcal{A}) \asymp \kappa_G m^{-\frac{3}{2}} \text{Area}(\mathcal{A})^2$. And,

$$\sum_{m \in \mathbb{N}} m^{-\frac{3}{2}} = \zeta\left(\frac{3}{2}\right) \approx 2.61238 < \infty.$$

So, by the theorem, the sum can not reach ∞ . \square

6. APPENDIX

6.1. The multiplicative asymptotic growth lemma. We will prove lemma 26 essentially by example. Our principle example is the famous Catalan numbers¹, which can be defined inductively by $c_0 = 1$ and

$$(23) \quad c_0 = 1 \quad \text{and} \quad c_{n+1} = \left(4 - \frac{6}{n+2}\right) c_n.$$

The numbers c_i are all integers. An alternate formula for them is $c_n = \frac{2n!}{n!2^{n+1}}$. It is well known that

$$(24) \quad c_n \asymp \frac{4^n}{\sqrt{\pi n^{3/2}}}.$$

¹The Catalan numbers are Sloane's sequence A000108. See <http://www.research.att.com/~njas/sequences/A000108>.

This can be derived from Stirling's formula, $n! \asymp \sqrt{2\pi n} \frac{n^n}{e^n}$. We will now give an interesting formula for the Catalan numbers, which may be new.

Proposition 32 (Catalan numbers). *The Catalan numbers are given by the formula*

$$c_n = \frac{4^{n-1}}{\pi} \int_{-\pi}^{\pi} (2 - 2 \cos \theta) \left(\frac{1 + \cos \theta}{2} \right)^n d\theta$$

Proof. The proof is essentially induction by repeated integration by parts. It is not hard to check the base case, that c_0 from the statement of the proposition is equal to one.

Let $x_n = \int_{-\pi}^{\pi} (2 - 2 \cos \theta) \left(\frac{1 + \cos \theta}{2} \right)^n d\theta$. Using the half-angle identities $\frac{1 - \cos \theta}{2} = \sin^2 \frac{\theta}{2}$ and $\frac{1 + \cos \theta}{2} = \cos^2 \frac{\theta}{2}$ and by making the substitution $\alpha = \frac{\theta}{2}$ we see $x_n = 8y_n$ where

$$y_n = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2 \alpha \cos^{2n} \alpha d\alpha.$$

We now apply integration by parts with $u = \cos^{2n} \alpha$, $du = -2n \sin \alpha \cos^{2n-1} \alpha$, $v = \frac{\alpha - \sin \alpha \cos \alpha}{2}$, and $dv = \sin^2 \alpha$ to obtain

$$(25) \quad y_n = \frac{\pi}{2} + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \alpha \sin \alpha \cos^{2n-1} \alpha d\alpha - ny_n$$

Again we apply integration by parts. This time we let $u = \cos^{2n-1} \alpha$, $du = -(2n-1) \sin \alpha \cos^{2n-2} \alpha$, $v = \sin \alpha - \alpha \cos \alpha$, and $dv = \alpha \sin \alpha$. We see

$$y_n = \frac{\pi}{2} - n\pi - n(2n-1) \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \alpha \sin \alpha \cos^{2n-1} \alpha d\alpha + n(2n-1)y_{n-1} - ny_n$$

But, this new integral appeared also in equation 25, so we may substitute yielding

$$y_n = \frac{\pi}{2} - n\pi - n(2n-1) \left(\frac{(n+1)y_n - \frac{\pi}{2}}{n} \right) + n(2n-1)y_{n-1} - ny_n$$

Then we solve for y_n in terms of y_{n-1} and see $y_n = \frac{2n-1}{2n+2}y_{n-1}$. Therefore, $x_n = \frac{2n-1}{2n+2}x_{n-1}$ as well. And with c_n as in the statement of the proposition, we see $c_n = \frac{4^{n-1}}{\pi}x_n$, so that

$$\frac{c_{n+1}}{c_n} = 4 \frac{x_{n+1}}{x_n} = 4 \frac{2n+1}{2n+4} = 4 - \frac{6}{2+n}$$

Thus, our numbers satisfy the recurrence relation of the Catalan numbers. See equation 23. \square

The following corollary is a special case of lemma 26, which proves it for one pair of functions for each $a > 0$.

Corollary 33. *For all constants $a > 0$ let f_a and g_a be defined*

$$f_a(\theta) = \frac{2 - 2 \cos(2\sqrt{a}\theta)}{4a} \quad \text{and} \quad g_a(\theta) = \frac{1 + \cos(2\sqrt{a}\theta)}{2}.$$

Let $0 < \epsilon < \frac{\pi}{\sqrt{a}}$. Then the sequence given by $r_n = \int_{-\epsilon}^{\epsilon} f_a(\theta)g_a(\theta)^n d\theta$ is asymptotic to $s_n = \frac{\sqrt{\pi}}{2a^{\frac{3}{2}}n^{\frac{3}{2}}}$.

Proof. We apply the change of variables $x = 2\sqrt{a}\theta$. We see

$$r_n = \frac{1}{8a^{\frac{3}{2}}} \int_{\frac{-\epsilon}{2\sqrt{a}}}^{\frac{\epsilon}{2\sqrt{a}}} (2 - 2 \cos x) \left(\frac{1 + \cos x}{2} \right)^n dx.$$

Temporarily, let $h_n(x) = (2 - 2 \cos x) \left(\frac{1 + \cos x}{2} \right)^n$. Then, we can write

$$(26) \quad r_n = \frac{1}{8a^{\frac{3}{2}}} \left(\int_{-\pi}^{\pi} h_n(x) dx - \int_{[-\pi, \frac{-\epsilon}{2\sqrt{a}}] \cup [\frac{\epsilon}{2\sqrt{a}}, \pi]} h_n(x) dx \right)$$

By proposition 32 and equation 24, the sequence

$$(27) \quad s_n = \frac{1}{8a^{\frac{3}{2}}} \int_{-\pi}^{\pi} h_n(x) dx = \frac{\pi}{2(4^n)a^{\frac{3}{2}}} c_n \asymp \frac{\sqrt{\pi}}{2n^{\frac{3}{2}}a^{\frac{3}{2}}}.$$

On the other hand, there is a constant $\delta < 1$ so that $\frac{1 + \cos x}{2} < \delta$ for all $x \in [-\pi, \frac{-\epsilon}{2\sqrt{a}}] \cup [\frac{\epsilon}{2\sqrt{a}}, \pi]$. Therefore

$$\left| \int_{[-\pi, \frac{-\epsilon}{2\sqrt{a}}] \cup [\frac{\epsilon}{2\sqrt{a}}, \pi]} h_n(x) dx \right| < \int_{[-\pi, \frac{-\epsilon}{2\sqrt{a}}] \cup [\frac{\epsilon}{2\sqrt{a}}, \pi]} 4\delta^n dx = 2\left(\pi - \frac{\epsilon}{2\sqrt{a}}\right)4\delta^n,$$

which is exponentially shrinking. Thus, equation 26 yields the asymptotic formula

$$\lim_{n \rightarrow \infty} \frac{r_n}{\left(\frac{\sqrt{\pi}}{2n^{\frac{3}{2}}a^{\frac{3}{2}}}\right)} = \lim_{n \rightarrow \infty} \frac{s_n}{\left(\frac{\sqrt{\pi}}{2n^{\frac{3}{2}}a^{\frac{3}{2}}}\right)} = 1.$$

□

Proof of Lemma 26. Let $a, \epsilon, f, g, r_n,$ and s_n be as in the statement of the lemma. We will show that

$$\lim_{n \rightarrow \infty} \frac{r_n}{n^{\frac{-3}{2}}} = \frac{\sqrt{\pi}}{2a^{\frac{3}{2}}}$$

using the squeeze theorem. We will provide upper and lower bounds for this limit converging to the stated limit.

Take any $a_+ > a$. Consider the functions $f_+(x) = \frac{1}{1+a_+-a} f_{a_+}(x)$ and $g_+(x) = g_{a_+}(x)$ with f_{a_+} and g_{a_+} as in corollary 33. Then there is an ϵ_+ with $0 < \epsilon_+ \leq \epsilon$ so that

- (1) $g_+(x) < g(x)$ for all $x \neq 0$ with $x \in [-\epsilon_+, \epsilon_+]$,
- (2) $f_+(x) < f(x)$ for all $x \neq 0$ with $x \in [-\epsilon_+, \epsilon_+]$.

Then it follows from corollary 33 that

$$\lim_{n \rightarrow \infty} \frac{r_n}{n^{\frac{-3}{2}}} > \lim_{n \rightarrow \infty} \frac{\int_{-\epsilon_+}^{\epsilon_+} f_+(x)g_+(x)^n dx}{n^{\frac{-3}{2}}} = \frac{\sqrt{\pi}}{2a_+^{\frac{3}{2}}(1 + a_+ - a)}.$$

Note that a_+ can be taken to be arbitrarily close to a yielding the inequality

$$\lim_{n \rightarrow \infty} \frac{r_n}{n^{\frac{-3}{2}}} \geq \frac{\sqrt{\pi}}{2a^{\frac{3}{2}}}.$$

A similar argument shows the opposite inequality. □

6.2. Bounds and derivatives of eigenvalues. The purpose of this section is to prove lemmas 29 and 28 of subsection 5.6. Further, this section implicitly relates the constant κ_G appearing in section 5.6 to billiards in hyperbolic triangles.

For a hyperbolic element G of the hyperbolic isometry group $PGL(2, \mathbb{R})$, the greatest eigenvalue λ has the geometric significance of

$$(28) \quad \inf_{x \in \mathbb{H}^2} dist(x, Gx) = 2 \log \lambda.$$

Moreover, this infimum is achieved. The collection of points where this infimum is achieved is a geodesic in \mathbb{H}^2 called the *axis* of the hyperbolic isometry G . The axis has a canonical orientation determined by the direction G translates the geodesic. If G belongs to a discrete group Γ , then this axis projects to a curve of finite length in the quotient \mathbb{H}^2/Γ .

In our case, Γ will be a triangle group generated by three reflections. See figure 3. The quotients $\Delta = \mathbb{H}^2/\Gamma$ will be triangular billiard tables. The projection of the axis of an element $G \in \Gamma$ will be a closed billiard path in the table \mathbb{H}^2/Γ . We can make this more concrete. Given G with eigenvalue $\lambda > 1$, consider an orientation and distance preserving map $\tilde{\gamma} : [0, 2 \log \lambda] \rightarrow Axis(G)$. This curve projects to the closed billiard path quotient $\gamma : [0, 2 \log \lambda] \rightarrow \mathbb{H}^2/\Gamma$, which bounces off edges according to the rules of optics.

We will need to extend these ideas to triangular billiard tables which are not quotients of \mathbb{H}^2 by a discrete group. Let Δ be any triangle in \mathbb{H}^2 with edges marked by the set $\{a, b, c\}$. We will use ℓ_a , ℓ_b , and ℓ_c to denote the bi-infinite geodesics in \mathbb{H}^2 that contain the marked edges. There is a natural representation to the isometry group of the hyperbolic plane, $\rho_\Delta : \mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2 \rightarrow PGL(2, \mathbb{R})$, determined by sending the generators to the reflections R_a , R_b , and R_c in the lines ℓ_a , ℓ_b , and

ℓ_c respectively. Given an element $G \in \mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2$, it may be written as a product of reflections

$$G = R_{e_n} R_{e_{n-1}} \dots R_{e_2} R_{e_1}.$$

We define the *orbit-class* $\Omega(G)$ to be the collection of all closed curves in \mathbb{H}^2 that visit the edges e_1, e_2, \dots, e_n in that order. We define the *orbit-length* of G to be $\ell(G) = \inf_{\gamma \in \Omega(G)} \text{length}(\gamma)$.

We have the following lemma.

Lemma 34. *Let Δ and G be as above. Let λ denote the eigenvalue of $\rho_\Delta(G)$ with largest eigenvalue. Then $2 \log |\lambda| \leq \ell(G)$.*

Proof. The statement is certainly true unless $\lambda > 1$. We will assume that there is a path $\gamma \in \Omega(G)$ with length less than $2 \log \lambda$ and draw a contradiction. We will construct a new path γ' in \mathbb{H}^2 with length equal to that of γ such that $\rho_\Delta(G)$ translates the starting point of γ' to the ending point of γ' . This will be a contradiction to equation 28.

We may assume γ begins on the geodesic ℓ_{e_1} and then travels to ℓ_{e_2} and so on. Let γ_i be the portion of γ which travels from ℓ_{e_i} to $\ell_{e_{i+1}}$, with γ_n the final arc of the path. Let $W_i = R_{e_i} R_{e_{i-1}} \dots R_{e_1}$ for $i = 1 \dots n$. We define γ' to be the concatenation of arcs

$$\gamma' = W_1(\gamma_1) \cup W_2(\gamma_2) \cup \dots \cup W_n(\gamma_n).$$

It can be seen by induction that this is a connected path. Since γ was closed and began on ℓ_{e_1} , the the fixed point set of W_1 , the transformation $G = W_n$ takes the starting point of γ' to the ending point of γ' as planned. And, γ has the same length as γ' . \square

We will need to introduce one more idea. The *Klein model* for the hyperbolic plane consists of $\mathbb{K}\mathbb{H}^2 = \{(x^2, y^2) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\}$. The boundary $\partial\mathbb{K}\mathbb{H}^2 = \{(x^2, y^2) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$. Geodesics in the Klein model are Euclidean line segments. Distance between points in the Klein model may be computed in two ways. Let P_1 and P_2 be two points in $\mathbb{K}\mathbb{H}^2$, and let $\overline{P_1 P_2}$ be the Euclidean line through them. Let Q_1 and Q_2 be the two points of $\partial\mathbb{K}\mathbb{H}^2 \cap \overline{P_1 P_2}$, chosen so that P_1 is closest in the Euclidean metric to Q_1 . Then the distance between P_1 and P_2 is given by the logarithm of the cross ratio,

$$(29) \quad \text{dist}_{\mathbb{K}\mathbb{H}^2}(P_1, P_2) = \log \left(\frac{\text{dist}_{\mathbb{R}^2}(P_1, P_2) \text{dist}_{\mathbb{R}^2}(Q_1, Q_2)}{\text{dist}_{\mathbb{R}^2}(P_1, Q_1) \text{dist}_{\mathbb{R}^2}(P_2, Q_2)} \right).$$

Alternately, we can compute distance by using the metric tensor ds .

$$(30) \quad ds = \sqrt{\frac{dx^2 + dy^2}{1 - x^2 - y^2} + \frac{(xdx + ydy)^2}{(1 - x^2 - y^2)^2}}.$$

The distance between two points can be computed by integrating the metric tensor over the geodesic path between them. See [CFKP97] or any hyperbolic geometry text for more details.

As discussed below theorem 1, the groups $\Gamma(S_c)$ are generated by the reflections A_c , B_c , and C_c in the sides of a triangle Δ_c in \mathbb{H}^2 together with $-I$, which acts trivially on \mathbb{H}^2 . Recall figure 3. When $c = \cos \theta \leq 1$, the triangle has two ideal vertices and one vertex with angle θ . This is true for all $c \leq 1$, not just the c of the form $\cos \frac{2\pi}{n}$. For our purposes, we will think of $\Delta_c \subset \mathbb{K}\mathbb{H}^2$. We define

$$(31) \quad \Delta_c = \text{Convex Hull}(\{P_1, P_2, P_3\}) \subset \mathbb{K}\mathbb{H}^2$$

where $P_1 = (-1, 0)$, $P_2 = (-1, 0)$, and $P_3 = (0, \frac{\sqrt{1+c}}{\sqrt{2}})$. This triangle is isometric to the triangle described for the group $\Gamma(S_c)$. The reflection lines of the elements A_c , B_c , and C_c are given by $\overline{P_3P_1}$, $\overline{P_1P_2}$, and $\overline{P_2P_3}$ respectively.

Proof of the eigenvalue upper bound lemma 29. Choose $\hat{G} \in \text{Aut}(S_{\geq 1})$ be chosen so that \hat{G} acts hyperbolically on S_1 . Then G_1 is a hyperbolic transformation of \mathbb{H}^2 . The projection of the axis of G_1 to $\mathbb{H}^2/\Gamma(S_1)$ minimizes length in the orbit-class $\Omega(G)$. In particular, this billiard path γ_1 realizes the infimum discussed in lemma 34.

Let λ_c denote the eigenvalue with greatest absolute value of $G_c \in \Gamma(S_c)$. We will use lemma 34 to show that $|\lambda_c| < \lambda_1$ for all $-1 \leq c < 1$. We define the linear map

$$(32) \quad M_c : \Delta_1 \rightarrow \Delta_c : (x, y) \mapsto (x, \frac{y\sqrt{1+c}}{\sqrt{2}}).$$

We claim that M_c shortens every line segment except line segments contained in the side $\overline{P_1P_2}$. Consider a segment \overline{XY} with finite length in Δ_1 . Let \overline{PQ} be the geodesic containing \overline{XY} , so that $P, Q \in \partial\mathbb{K}\mathbb{H}^2$ with P the closest to X as in the left side of figure 11. Then by equation 29,

$$\begin{aligned} \text{dist}_{\mathbb{K}\mathbb{H}^2}(X, Y) &= \log \left(\frac{\text{dist}_{\mathbb{R}^2}(X, Y) \text{dist}_{\mathbb{R}^2}(P, Q)}{\text{dist}_{\mathbb{R}^2}(X, P) \text{dist}_{\mathbb{R}^2}(Y, Q)} \right) \\ &= \log \left(\frac{\text{dist}_{\mathbb{R}^2}(M_c(X), M_c(Y)) \text{dist}_{\mathbb{R}^2}(M_c(P), M_c(Q))}{\text{dist}_{\mathbb{R}^2}(M_c(X), M_c(P)) \text{dist}_{\mathbb{R}^2}(M_c(Y), M_c(Q))} \right). \end{aligned}$$

And, let $P', Q' \in \partial\mathbb{K}\mathbb{H}^2$ be the points where the geodesic $\overline{M_c(P)M_c(Q)}$ intersects the boundary. Then,

$$\text{dist}_{\mathbb{K}\mathbb{H}^2}(M_c(X), M_c(Y)) = \log \left(\frac{\text{dist}_{\mathbb{R}^2}(M_c(X), M_c(Y)) \text{dist}_{\mathbb{R}^2}(P', Q')}{\text{dist}_{\mathbb{R}^2}(M_c(X), P') \text{dist}_{\mathbb{R}^2}(M_c(Y), Q')} \right).$$

It is a standard computation that $\text{dist}_{\mathbb{K}\mathbb{H}^2}(X, Y) > \text{dist}_{\mathbb{K}\mathbb{H}^2}(M_c(X), M_c(Y))$ so long as either $M_c(P) \neq P'$ or $M_c(Q) \neq Q'$. In particular, the only time this inequality could be false is when $PQ \subset \{(x, y) | y = 0\}$. Our claim is proved.

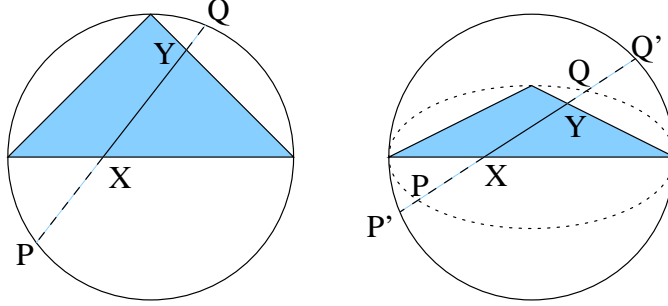


FIGURE 11. A segment and its image under M_c .

No finite length billiard path can have a segment contained in the line $y = 0$, therefore

$$\text{length}(M_c(\gamma_1)) < \text{length}(\gamma_1).$$

whenever $-1 \leq c < 1$. Then by lemma 34, for all such c ,

$$2 \log |\lambda_c| \leq \text{length}(M_c(\gamma_1)) < \text{length}(\gamma_1) = 2 \log \lambda_1.$$

Thus $|\lambda_c| < \lambda_1$. □

Proof of the derivative of the eigenvalue lemma 28. Let $\hat{G} \in \text{Aut}(S_{\geq 1})$ be chosen so that \hat{G} acts hyperbolically on S_1 . Let λ_c be the largest eigenvalue of G_c . We must show

$$\left. \frac{d}{dc} \lambda_c \right|_{c=1} > 0.$$

Clearly $\frac{d}{dc} \lambda_c$ exists, since entries of G_c vary polynomially in c . We can afford to restrict attention to the case of $c < 1$, so we may recall the map $M_c : \Delta_1 \rightarrow \Delta_c$ of equation 32.

Let $m = \frac{\sqrt{1+c}}{\sqrt{2}}$. We have $\frac{d}{dc} m = \frac{1}{2\sqrt{2+2c}}$, and $\frac{d}{dc} m|_{c=1} = \frac{1}{4}$. By equation 30, the $\mathbb{K}\mathbb{H}^2$ length of the vector $\mathbf{i} = (1, 0)$ at the point $(x, y) \in \mathbb{K}\mathbb{H}$ is given by

$$I_1 = \frac{\sqrt{1-y^2}}{1-x^2-y^2}.$$

The length of the vector $M_c(\mathbf{i}) = (1, 0)$ at the point $M_c(x, y) = (x, my)$ is given by

$$I_2 = \frac{\sqrt{1 - m^2 y^2}}{1 - x^2 - m^2 y^2}.$$

We compute

$$\frac{d}{dc} \left[\frac{I_2}{I_1} \right]_{c=1} = \frac{y^2(1 + x^2 - y^2)}{4(1 - y^2)(1 - x^2 - y^2)} > 0.$$

Note, we are perturbing c in the negative direction. This says that off the line $y = 0$, M_c is compressing every horizontal vector enough to be detected by the first derivative. Let J_1 be the $\mathbb{K}\mathbb{H}^2$ length of the vector $\mathbf{j} = (0, 1)$ at the point $(x, y) \in \mathbb{K}\mathbb{H}$ and J_2 be the $\mathbb{K}\mathbb{H}^2$ length of the vector $M_c(\mathbf{j}) = (0, m)$ at the point $M_c(x, y) = (x, my)$. We have

$$J_1 = \frac{\sqrt{1 - x^2}}{1 - x^2 - y^2} \quad \text{and} \quad J_2 = \frac{m\sqrt{1 - x^2}}{1 - x^2 - m^2 y^2}.$$

We compute

$$\frac{d}{dc} \left[\frac{J_2}{J_1} \right]_{c=1} = \frac{1 - x^2 + y^2}{4(1 - x^2 - y^2)} > 0.$$

In this case, M_c is compressing every vertical vector enough to be detected by the first derivative.

The argument concludes in the same manner as the previous proof. Let γ_1 be the billiard path on Δ_1 corresponding to G_1 . The argument above tells us that $\frac{d}{dc} \text{length}(M_c(\gamma_1)) = k > 0$. But for $c < 1$,

$$2 \log \lambda_c \leq \text{length}(M_c(\gamma_1)) = \text{length}(\gamma_1) - k(1 - c) + \text{higher order terms}.$$

By taking a straight forward derivative, we get

$$2 \log \lambda_c = 2 \log \lambda_1 - (1 - c) \frac{2 \frac{d}{dc} \lambda_c|_{c=1}}{\lambda_1} + \text{higher order terms}.$$

This pair of equations imply

$$\frac{d}{dc} \lambda_c|_{c=1} \geq \frac{k \lambda_1}{2} > 0.$$

□

REFERENCES

- [CFKP97] James W. Cannon, William J. Floyd, Richard Kenyon, and Walter R. Parry, *Hyperbolic geometry*, Flavors of geometry, Math. Sci. Res. Inst. Publ., vol. 31, Cambridge Univ. Press, Cambridge, 1997, pp. 59–115. MR1491098 (99c:57036)

- [GT96] Eugene Gutkin and Serge Troubetzkoy, *Directional flows and strong recurrence for polygonal billiards*, International Conference on Dynamical Systems (Montevideo, 1995), Pitman Res. Notes Math. Ser., vol. 362, Longman, Harlow, 1996, pp. 21–45. MR1460795 (98h:58104)
- [Khi97] A. Ya. Khinchin, *Continued fractions*, russian ed., Dover Publications Inc., Mineola, NY, 1997, With a preface by B. V. Gnedenko, Reprint of the 1964 translation. MR1451873 (98c:11008)
- [MT02] Howard Masur and Serge Tabachnikov, *Rational billiards and flat structures*, Handbook of dynamical systems, Vol. 1A, North-Holland, Amsterdam, 2002, pp. 1015–1089. MR1451873 (98c:11008)
- [RS92] Andrew M. Rockett and Peter Szűsz, *Continued fractions*, World Scientific Publishing Co. Inc., River Edge, NJ, 1992. MR1188878 (93m:11060)
- [Thu97] William P. Thurston, *Three-dimensional geometry and topology. Vol. 1*, Princeton Mathematical Series, vol. 35, Princeton University Press, Princeton, NJ, 1997, Edited by Silvio Levy. MR1435975 (97m:57016)
- [Vee89] W. A. Veech, *Teichmüller curves in moduli space, Eisenstein series and an application to triangular billiards*, Invent. Math. **97** (1989), no. 3, 553–583. MR1435975 (97m:57016)